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Electromagnetic Field Theory: A Problem Solving Approach

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Cartesian Coordinates (x, y, z)

$$\nabla f = \frac{\partial f}{\partial x} \mathbf{i}_x + \frac{\partial f}{\partial y} \mathbf{i}_y + \frac{\partial f}{\partial z} \mathbf{i}_z$$

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \mathbf{A} = \mathbf{i}_x \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + \mathbf{i}_y \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + \mathbf{i}_z \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right)$$

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

Cylindrical Coordinates (r, ϕ, z)

$$\nabla f = \frac{\partial f}{\partial r} \mathbf{i}_r + \frac{1}{r} \frac{\partial f}{\partial \phi} \mathbf{i}_\phi + \frac{\partial f}{\partial z} \mathbf{i}_z$$

$$\nabla \cdot \mathbf{A} = \frac{1}{r} \frac{\partial}{\partial r} (r A_r) + \frac{1}{r} \frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \mathbf{A} = \mathbf{i}_r \left(\frac{1}{r} \frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z} \right) + \mathbf{i}_\phi \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} \right) + \mathbf{i}_z \left[\frac{1}{r} \left(\frac{\partial (r A_\phi)}{\partial r} - \frac{\partial A_r}{\partial \phi} \right) \right]$$

$$\nabla^2 f = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial f}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}$$

Spherical Coordinates (r, θ, ϕ)

$$\nabla f = \frac{\partial f}{\partial r} \mathbf{i}_r + \frac{1}{r} \frac{\partial f}{\partial \theta} \mathbf{i}_\theta + \frac{1}{r \sin \theta} \frac{\partial f}{\partial \phi} \mathbf{i}_\phi$$

$$\nabla \cdot \mathbf{A} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 A_r) + \frac{1}{r \sin \theta} \frac{\partial (\sin \theta A_\theta)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial A_\phi}{\partial \phi}$$

$$\nabla \times \mathbf{A} = \mathbf{i}_r \frac{1}{r \sin \theta} \left[\frac{\partial (\sin \theta A_\phi)}{\partial \theta} - \frac{\partial A_\theta}{\partial \phi} \right] + \mathbf{i}_\theta \frac{1}{r} \left[\frac{1}{\sin \theta} \frac{\partial A_r}{\partial \phi} - \frac{\partial (r A_\phi)}{\partial r} \right] + \mathbf{i}_\phi \frac{1}{r} \left[\frac{\partial (r A_\theta)}{\partial r} - \frac{\partial A_r}{\partial \theta} \right]$$

$$\nabla^2 f = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 f}{\partial \phi^2}$$

<i>Cartesian</i>		<i>Cylindrical</i>		<i>Spherical</i>
x	=	$r \cos \phi$	=	$r \sin \theta \cos \phi$
y	=	$r \sin \phi$	=	$r \sin \theta \sin \phi$
z	=	z	=	$r \cos \theta$
\mathbf{i}_x	=	$\cos \phi \mathbf{i}_r - \sin \phi \mathbf{i}_\phi$	=	$\sin \theta \cos \phi \mathbf{i}_r + \cos \theta \cos \phi \mathbf{i}_\theta - \sin \phi \mathbf{i}_\phi$
\mathbf{i}_y	=	$\sin \phi \mathbf{i}_r + \cos \phi \mathbf{i}_\phi$	=	$\sin \theta \sin \phi \mathbf{i}_r + \cos \theta \sin \phi \mathbf{i}_\theta + \cos \phi \mathbf{i}_\phi$
\mathbf{i}_z	=	\mathbf{i}_z	=	$\cos \theta \mathbf{i}_r - \sin \theta \mathbf{i}_\theta$

<i>Cylindrical</i>		<i>Cartesian</i>		<i>Spherical</i>
r	=	$\sqrt{x^2 + y^2}$	=	$r \sin \theta$
ϕ	=	$\tan^{-1} y/x$	=	ϕ
z	=	z	=	$r \cos \theta$
\mathbf{i}_r	=	$\cos \phi \mathbf{i}_x + \sin \phi \mathbf{i}_y$	=	$\sin \theta \mathbf{i}_r + \cos \theta \mathbf{i}_\theta$
\mathbf{i}_ϕ	=	$-\sin \phi \mathbf{i}_x + \cos \phi \mathbf{i}_y$	=	\mathbf{i}_ϕ
\mathbf{i}_z	=	\mathbf{i}_z	=	$\cos \theta \mathbf{i}_r - \sin \theta \mathbf{i}_\theta$

<i>Spherical</i>		<i>Cartesian</i>		<i>Cylindrical</i>
r	=	$\sqrt{x^2 + y^2 + z^2}$	=	$\sqrt{r^2 + z^2}$
θ	=	$\cos^{-1} \frac{z}{\sqrt{x^2 + y^2 + z^2}}$	=	$\cos^{-1} \frac{z}{\sqrt{r^2 + z^2}}$
ϕ	=	$\cot^{-1} x/y$	=	ϕ
\mathbf{i}_r	=	$\sin \theta \cos \phi \mathbf{i}_x + \sin \theta \sin \phi \mathbf{i}_y + \cos \theta \mathbf{i}_z$	=	$\sin \theta \mathbf{i}_r + \cos \theta \mathbf{i}_z$
\mathbf{i}_θ	=	$\cos \theta \cos \phi \mathbf{i}_x + \cos \theta \sin \phi \mathbf{i}_y - \sin \theta \mathbf{i}_z$	=	$\cos \theta \mathbf{i}_r - \sin \theta \mathbf{i}_z$
\mathbf{i}_ϕ	=	$-\sin \phi \mathbf{i}_x + \cos \phi \mathbf{i}_y$	=	\mathbf{i}_ϕ

Geometric relations between coordinates and unit vectors for Cartesian, cylindrical, and spherical coordinate systems.

VECTOR IDENTITIES

$$(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C} = \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = (\mathbf{C} \times \mathbf{A}) \cdot \mathbf{B}$$

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$$

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0$$

$$\nabla \times (\nabla f) = 0$$

$$\nabla(fg) = f\nabla g + g\nabla f$$

$$\begin{aligned} \nabla(\mathbf{A} \cdot \mathbf{B}) &= (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} \\ &\quad + \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) \end{aligned}$$

$$\nabla \cdot (f\mathbf{A}) = f\nabla \cdot \mathbf{A} + (\mathbf{A} \cdot \nabla)f$$

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$$

$$\begin{aligned} \nabla \times (\mathbf{A} \times \mathbf{B}) &= \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) \\ &\quad + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} \end{aligned}$$

$$\nabla \times (f\mathbf{A}) = \nabla f \times \mathbf{A} + f\nabla \times \mathbf{A}$$

$$(\nabla \times \mathbf{A}) \times \mathbf{A} = (\mathbf{A} \cdot \nabla)\mathbf{A} - \frac{1}{2}\nabla(\mathbf{A} \cdot \mathbf{A})$$

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$$

INTEGRAL THEOREMS

Line Integral of a Gradient

$$\int_a^b \nabla f \cdot d\mathbf{l} = f(b) - f(a)$$

Divergence Theorem:

$$\int_V \nabla \cdot \mathbf{A} dV = \oint_S \mathbf{A} \cdot d\mathbf{S}$$

Corollaries

$$\int_V \nabla f dV = \oint_S f d\mathbf{S}$$

$$\int_V \nabla \times \mathbf{A} dV = -\oint_S \mathbf{A} \times d\mathbf{S}$$

Stokes' Theorem:

$$\oint_L \mathbf{A} \cdot d\mathbf{l} = \int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S}$$

Corollary

$$\oint_L f d\mathbf{l} = -\int_S \nabla f \times d\mathbf{S}$$

MAXWELL'S EQUATIONS

<i>Integral</i>	<i>Differential</i>	<i>Boundary Conditions</i>
Faraday's Law		
$\oint_L \mathbf{E}' \cdot d\mathbf{l} = -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{S}$	$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	$\mathbf{n} \times (\mathbf{E}'_2 - \mathbf{E}'_1) = 0$
Ampere's Law with Maxwell's Displacement Current Correction		
$\oint_L \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J}_f \cdot d\mathbf{S} + \frac{d}{dt} \int_S \mathbf{D} \cdot d\mathbf{S}$	$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}$	$\mathbf{n} \times (\mathbf{H}_2 - \mathbf{H}_1) = \mathbf{K}_f$

Gauss's Law

$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_V \rho_f dV$	$\nabla \cdot \mathbf{D} = \rho_f$	$\mathbf{n} \cdot (\mathbf{D}_2 - \mathbf{D}_1) = \sigma_f$
$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0$	$\nabla \cdot \mathbf{B} = 0$	$\mathbf{n} \cdot (\mathbf{B}_2 - \mathbf{B}_1) = 0$

Conservation of Charge

$\oint_S \mathbf{J}_f \cdot d\mathbf{S} + \frac{d}{dt} \int_V \rho_f dV = 0$	$\nabla \cdot \mathbf{J}_f + \frac{\partial \rho_f}{\partial t} = 0$	$\mathbf{n} \cdot (\mathbf{J}_2 - \mathbf{J}_1) + \frac{\partial \sigma_f}{\partial t} = 0$
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Usual Linear Constitutive Laws

$$\mathbf{D} = \epsilon \mathbf{E}$$

$$\mathbf{B} = \mu \mathbf{H}$$

$$\mathbf{J}_f = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B}) = \sigma \mathbf{E}' \text{ [Ohm's law for moving media with velocity } \mathbf{v}]$$

PHYSICAL CONSTANTS

Constant	Symbol	Value	units
Speed of light in vacuum	c	$2.9979 \times 10^8 \approx 3 \times 10^8$	m/sec
Elementary electron charge	e	1.602×10^{-19}	coul
Electron rest mass	m_e	9.11×10^{-31}	kg
Electron charge to mass ratio	$\frac{e}{m_e}$	1.76×10^{11}	coul/kg
Proton rest mass	m_p	1.67×10^{-27}	kg
Boltzmann constant	k	1.38×10^{-23}	joule/°K
Gravitation constant	G	6.67×10^{-11}	nt-m ² /(kg) ²
Acceleration of gravity	g	9.807	m/(sec) ²
Permittivity of free space	ϵ_0	$8.854 \times 10^{-12} \approx \frac{10^{-9}}{36\pi}$	farad/m
Permeability of free space	μ_0	$4\pi \times 10^{-7}$	henry/m
Planck's constant	h	6.6256×10^{-34}	joule-sec
Impedance of free space	$\eta_0 = \sqrt{\frac{\mu_0}{\epsilon_0}}$	$376.73 \approx 120\pi$	ohms
Avogadro's number	N	6.023×10^{23}	atoms/mole