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Electromagnetic Field Theory: A Problem Solving Approach

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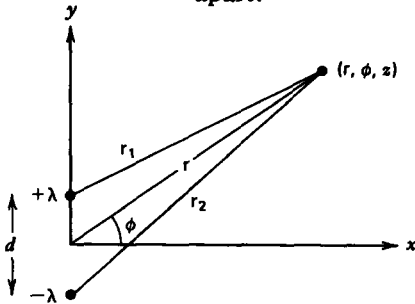
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PROBLEMS

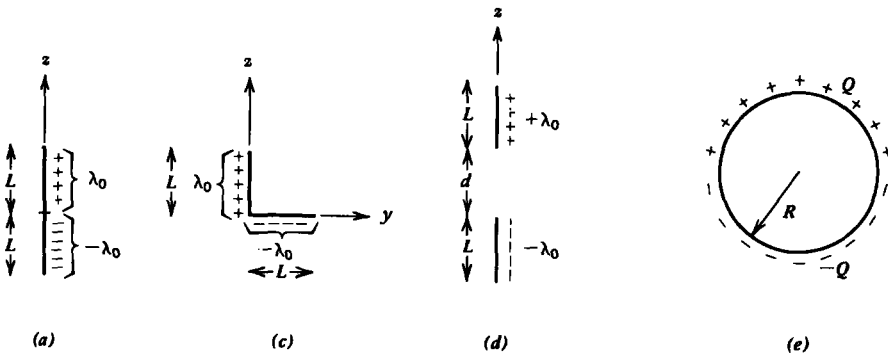
Section 3-1

1. A two-dimensional dipole is formed by two infinitely long parallel line charges of opposite polarity $\pm\lambda$ a small distance d , apart.



- (a) What is the potential at any coordinate (r, ϕ, z) ?
- (b) What are the potential and electric field far from the dipole ($r \gg d$)? What is the dipole moment per unit length?
- (c) What is the equation of the field lines?

2. Find the dipole moment for each of the following charge distributions:



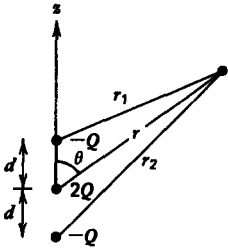
- (a) Two uniform colinear opposite polarity line charges $\pm\lambda_0$ each a small distance L along the z axis.
- (b) Same as (a) with the line charge distribution as

$$\lambda(z) = \begin{cases} \lambda_0(1 - z/L), & 0 < z < L \\ -\lambda_0(1 + z/L), & -L < z < 0 \end{cases}$$

- (c) Two uniform opposite polarity line charges $\pm\lambda_0$ each of length L but at right angles.
- (d) Two parallel uniform opposite polarity line charges $\pm\lambda_0$ each of length L a distance $d\hat{i}_z$ apart.

- (e) A spherical shell with total uniformly distributed surface charge Q on the upper half and $-Q$ on the lower half. (Hint: $\mathbf{i}_r = \sin \theta \cos \phi \mathbf{i}_x + \sin \theta \sin \phi \mathbf{i}_y + \cos \theta \mathbf{i}_z$.)
- (f) A spherical volume with total uniformly distributed volume charge of Q in the upper half and $-Q$ on the lower half. (Hint: Integrate the results of (e).)

3. The linear quadrupole consists of two superposed dipoles along the z axis. Find the potential and electric field for distances far away from the charges ($r \gg d$).



Linear quadrupole

$$\frac{1}{r_1} \approx \frac{1}{r} \left\{ 1 + \frac{d}{r} \cos \theta - \frac{1}{2} \left(\frac{d}{r}\right)^2 (1 - 3 \cos^2 \theta) \right\}$$

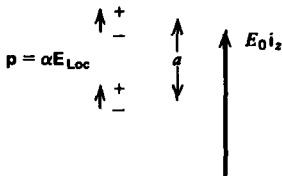
$$\frac{1}{r_2} \approx \frac{1}{r} \left\{ 1 - \frac{d}{r} \cos \theta - \frac{1}{2} \left(\frac{d}{r}\right)^2 (1 - 3 \cos^2 \theta) \right\}$$

4. Model an atom as a fixed positive nucleus of charge Q with a surrounding spherical negative electron cloud of nonuniform charge density:

$$\rho = -\rho_0(1 - r/R_0), \quad r < R_0$$

- (a) If the atom is neutral, what is ρ_0 ?
- (b) An electric field is applied with local field \mathbf{E}_{Loc} causing a slight shift \mathbf{d} between the center of the spherical cloud and the positive nucleus. What is the equilibrium dipole spacing?
- (c) What is the approximate polarizability α if $9\epsilon_0 E_{Loc}/(\rho_0 R_0) \ll 1$?

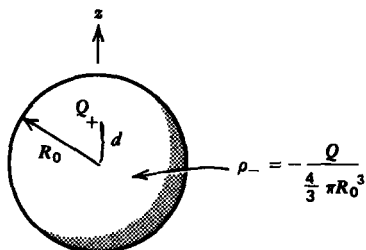
5. Two colinear dipoles with polarizability α are a distance a apart along the z axis. A uniform field $E_0 \mathbf{i}_z$ is applied.



- (a) What is the total local field seen by each dipole?
- (b) Repeat (a) if we have an infinite array of dipoles with constant spacing a . (Hint: $\sum_{n=1}^{\infty} 1/n^3 \approx 1.2$.)
- (c) If we assume that we have one such dipole within each volume of a^3 , what is the permittivity of the medium?

6. A dipole is modeled as a point charge Q surrounded by a spherical cloud of electrons with radius R_0 . Then the local

field \mathbf{E}_{Loc} differs from the applied field \mathbf{E} by the field due to the dipole itself. Since \mathbf{E}_{dip} varies within the spherical cloud, we use the average field within the sphere.



(a) Using the center of the cloud as the origin, show that the dipole electric field within the cloud is

$$\mathbf{E}_{\text{dip}} = -\frac{Qr\mathbf{i}_r}{4\pi\epsilon_0R_0^3} + \frac{Q(\mathbf{i}_r - d\mathbf{i}_z)}{4\pi\epsilon_0[d^2 + r^2 - 2rd \cos \theta]^{3/2}}$$

(b) Show that the average x and y field components are zero. (Hint: $\mathbf{i}_r = \sin \theta \cos \phi \mathbf{i}_x + \sin \theta \sin \phi \mathbf{i}_y + \cos \theta \mathbf{i}_z$.)

(c) What is the average z component of the field? (Hint: Change variables to $u = r^2 + d^2 - 2rd \cos \theta$ and remember $\sqrt{(r-d)^2} = |r-d|$.)

(d) If we have one dipole within every volume of $\frac{4}{3}\pi R_0^3$, how is the polarization \mathbf{P} related to the applied field \mathbf{E} ?

7. Assume that in the dipole model of Figure 3-5a the mass of the positive charge is so large that only the electron cloud moves as a solid mass m .

(a) The local electric field is \mathbf{E}_0 . What is the dipole spacing?

(b) At $t = 0$, the local field is turned off ($\mathbf{E}_0 = 0$). What is the subsequent motion of the electron cloud?

(c) What is the oscillation frequency if Q has the charge and mass of an electron with $R_0 = 10^{-10}$ m?

(d) In a real system there is always some damping that we take to be proportional to the velocity ($\mathbf{f}_{\text{damping}} = -\beta\mathbf{v}$). What is the equation of motion of the electron cloud for a sinusoidal electric field $\text{Re}(\hat{\mathbf{E}}_0 e^{j\omega t})$?

(e) Writing the driven displacement of the dipole as

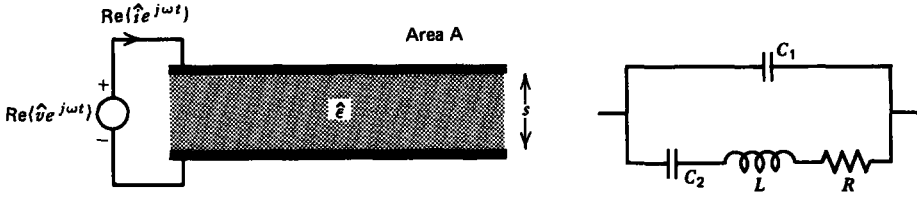
$$d = \text{Re}(\hat{d} e^{j\omega t}).$$

what is the complex polarizability \hat{a} , where $\hat{p} = Q\hat{d} = \hat{a}\hat{\mathbf{E}}_0$?

(f) What is the complex dielectric constant $\hat{\epsilon} = \epsilon_r + j\epsilon_i$ of the system? (Hint: Define $\omega_p^2 = Q^2 N / (m\epsilon_0)$.)

(g) Such a dielectric is placed between parallel plate electrodes. Show that the equivalent circuit is a series R, L, C shunted by a capacitor. What are C_1, C_2, L , and R ?

(h) Consider the limit where the electron cloud has no mass ($m = 0$). With the frequency ω as a parameter show that

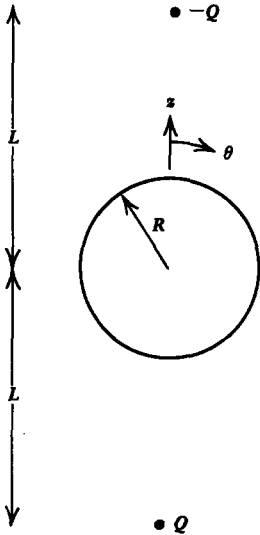


(g)

a plot of ϵ_r versus ϵ_i is a circle. Where is the center of the circle and what is its radius? Such a diagram is called a Cole-Cole plot.

(i) What is the maximum value of ϵ_i and at what frequency does it occur?

8. Two point charges of opposite sign $\pm Q$ are a distance L above and below the center of a grounded conducting sphere of radius R .



(a) What is the electric field everywhere along the z axis and in the $\theta = \pi/2$ plane? (**Hint:** Use the method of images.)

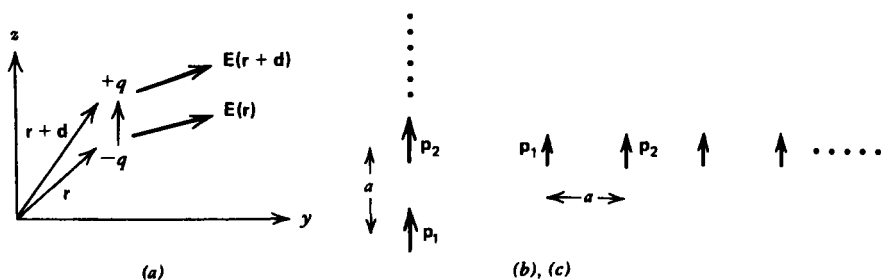
(b) We would like this problem to model the case of a conducting sphere in a uniform electric field by bringing the point charges $\pm Q$ out to infinity ($L \rightarrow \infty$). What must the ratio Q/L^2 be such that the field far from the sphere in the $\theta = \pi/2$ plane is $E_0 \hat{i}_x$?

(c) In this limit, what is the electric field everywhere?

9. A dipole with moment \mathbf{p} is placed in a nonuniform electric field.

(a) Show that the force on a dipole is

$$\mathbf{f} = (\mathbf{p} \cdot \nabla)\mathbf{E}$$

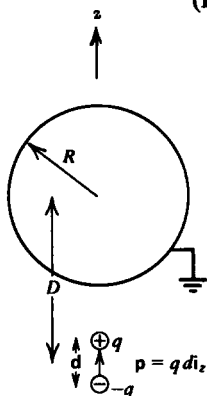


(b) Find the force on dipole 1 due to dipole 2 when the two dipoles are colinear, or are adjacent a distance a apart.

(c) Find the force on dipole 1 if it is the last dipole in an infinite array of identical colinear or adjacent dipoles with spacing a . (Hint: $\sum_{n=1}^{\infty} 1/n^4 = \pi^4/90$.)

10. A point dipole with moment $p\mathbf{i}_z$ is a distance D from the center of a grounded sphere of radius R .

(Hint: $d \ll D$.)



- What is the induced dipole moment of the sphere?
- What is the electric field everywhere along the z axis?
- What is the force on the sphere? (Hint: See Problem 9a.)

Section 3-2

11. Find the potential, electric field, and charge density distributions for each of the following charges placed within a medium of infinite extent, described by drift-diffusion conduction in the limit when the electrical potential is much less than the thermal voltage ($qV/kT \ll 1$):

- Sheet of surface charge σ_f placed at $x = 0$.
- Infinitely long line charge with uniform density λ . (Hint: Bessel's equation results.)
- Conducting sphere of radius R carrying a total surface charge Q .

12. Two electrodes at potential $\pm V_0/2$ located at $x = \pm l$ enclose a medium described by drift-diffusion conduction for two oppositely charged carriers, where $qV_0/kT \ll 1$.

(a) Find the approximate solutions of the potential, electric field, and charge density distributions. What is the charge polarity near each electrode?

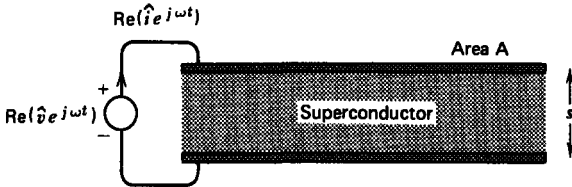
(b) What is the total charge per unit area within the volume of the medium and on each electrode?

13. (a) Neglecting diffusion effects but including charge inertia and collisions, what is the time dependence of the velocity of charge carriers when an electric field $E_0 \hat{i}_x$ is instantaneously turned on at $t = 0$?

(b) After the charge carriers have reached their steady-state velocity, the electric field is suddenly turned off. What is their resulting velocity?

(c) This material is now placed between parallel plate electrodes of area A and spacing s . A sinusoidal voltage is applied $\text{Re}(V_0 e^{j\omega t})$. What is the equivalent circuit?

14. Parallel plate electrodes enclose a superconductor that only has free electrons with plasma frequency ω_{pe} .



(a) What is the terminal current when a sinusoidal voltage is applied?

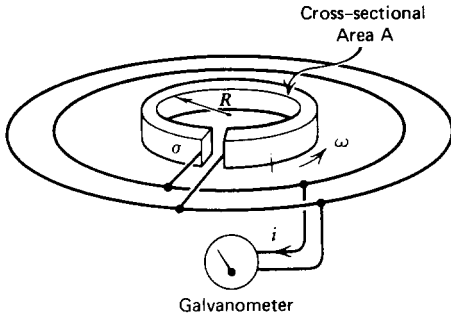
(b) What is the equivalent circuit?

15. A conducting ring of radius R is rotated at constant angular speed. The ring has Ohmic conductivity σ and cross sectional area A . A galvanometer is connected to the ends of the ring to indicate the passage of any charge. The connection is made by slip rings so that the rotation of the ring is unaffected by the galvanometer. The ring is instantly stopped, but the electrons within the ring continue to move a short time until their momentum is dissipated by collisions. For a particular electron of charge q and mass m conservation of momentum requires

$$\Delta(mv) = \int F dt$$

where $F = qE$ is the force on the electron.

(a) For the Ohmic conductor, relate the electric field to the current in the wire.

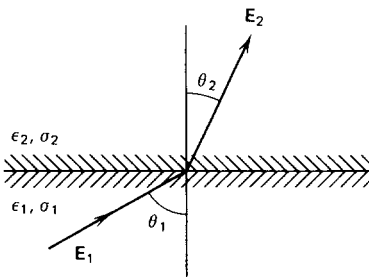


(b) When the ring is instantly stopped, what is the charge Q through the galvanometer? (**Hint:** $Q = \int i dt$. This experiment is described by R. C. Tolman and T. D. Stewart, *Phys. Rev.* **8**, No. 2 (1916), p. 97.)

(c) If the ring is an electron superconductor with plasma frequency ω_p , what is the resulting current in the loop when it stops?

Section 3.3

16. An electric field with magnitude E_1 is incident upon the interface between two materials at angle θ_1 from the normal. For each of the following material properties find the magnitude and direction of the field \mathbf{E}_2 in region 2.



(a) Lossless dielectrics with respective permittivities ϵ_1 and ϵ_2 . There is no interfacial surface charge.

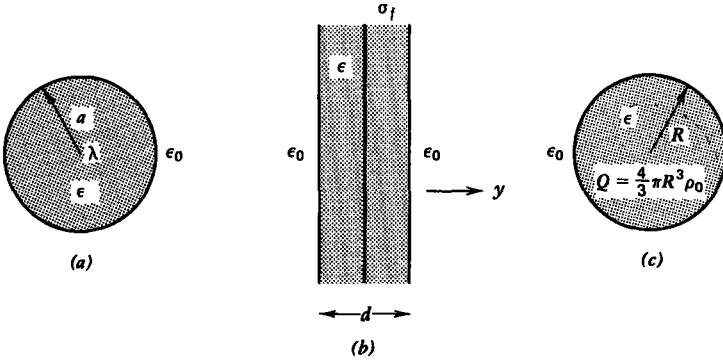
(b) Ohmic materials with respective conductivities σ_1 and σ_2 in the dc steady state. What is the free surface charge density σ_f on the interface?

(c) Lossy dielectrics (ϵ_1, σ_1) and (ϵ_2, σ_2) with a sinusoidally varying electric field

$$E_1 = \text{Re} (\hat{E}_1 e^{j\omega t})$$

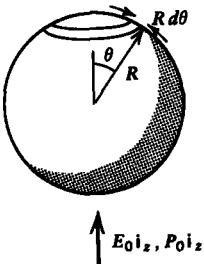
What is the free surface charge density σ_f on the interface?

17. Find the electric, displacement, and polarization fields and the polarization charge everywhere for each of the following configurations:



- (a) An infinitely long line charge λ placed at the center of a dielectric cylinder of radius a and permittivity ϵ .
- (b) A sheet of surface charge σ_f placed at the center of a dielectric slab with permittivity ϵ and thickness d .
- (c) A uniformly charged dielectric sphere with permittivity ϵ and radius R carrying a total free charge Q .

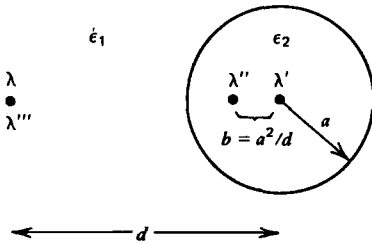
18. Lorentz calculated the local field acting on a dipole due to a surrounding uniformly polarized medium stressed by a macroscopic field $E_0 \mathbf{i}_z$ by encircling the dipole with a small spherical free space cavity of radius R .



- (a) If the medium outside the cavity has polarization $P_0 \mathbf{i}_z$, what is the surface polarization charge on the spherical interface? (Hint: $\mathbf{i}_z = \mathbf{i}_r \cos \theta - \mathbf{i}_\theta \sin \theta$)
- (b) Break this surface polarization charge distribution into hoop line charge elements of thickness $d\theta$. What is the total charge on a particular shell at angle θ ?
- (c) What is the electric field due to this shell at the center of the sphere where the dipole is?
- (d) By integrating over all shells, find the total electric field acting on the dipole. This is called the Lorentz field. (Hint: Let $u = \cos \theta$).

19. A line charge λ within a medium of permittivity ϵ_1 is outside a dielectric cylinder of radius a and permittivity ϵ_2 .

The line charge is parallel to the cylinder axis and a distance d from it.



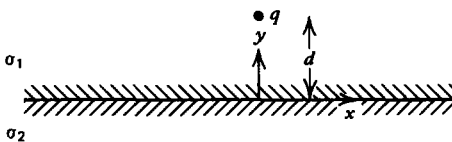
(a) Try using the method of images by placing a line charge λ' at the center and another image λ'' within the cylinder at distance $b = a^2/d$ from the axis along the line joining the axis to the line charge. These image charges together with the original line charge will determine the electric field outside the cylinder. Put another line charge λ''' at the position of the original line charge to determine the field within the cylinder. What values of λ' , λ'' , and λ''' satisfy the boundary conditions?

(b) Check your answers with that of Section 3-3-3 in the limit as the radius of the cylinder becomes large so that it looks like a plane.

(c) What is the force per unit length on the line charge λ ?

(d) Repeat (a)–(c) when the line charge λ is within the dielectric cylinder.

20. A point charge q is a distance d above a planar boundary separating two Ohmic materials with respective conductivities σ_1 and σ_2 .



(a) What steady-state boundary conditions must the electric field satisfy?

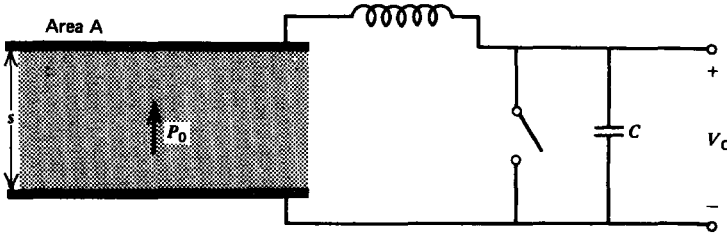
(b) What image charge configuration will satisfy these boundary conditions? (**Hint:** See Section 3-3-3.)

(c) What is the force on q ?

21. The polarization of an electret is measured by placing it between parallel plate electrodes that are shorted together.

(a) What is the surface charge on the upper electrode?

(b) The switch is then opened and the upper electrode is taken far away from the electret. What voltage is measured across the capacitor?

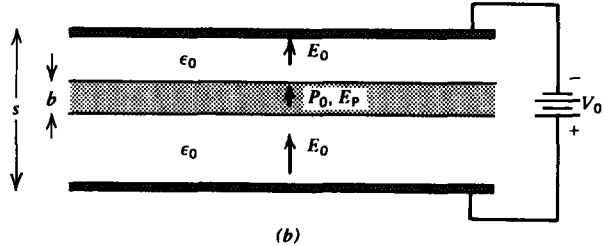
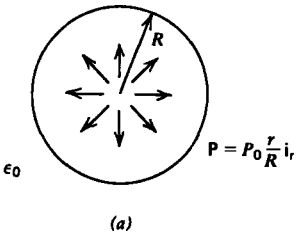


22. A cylinder of radius a and height L as in Figure 3-14, has polarization

$$\mathbf{P} = \frac{P_0 z}{L} \mathbf{i}_z$$

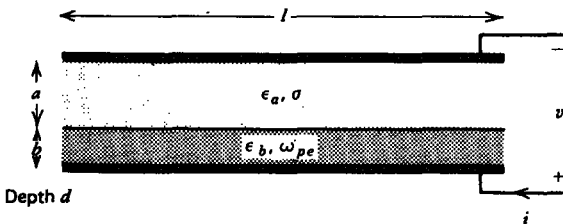
- (a) What is the polarization charge distribution?
- (b) Find the electric and displacement fields everywhere along the z axis. (**Hint:** Use the results of Sections 2-3-5b and 2-3-5d.)

23. Find the electric field everywhere for the following permanently polarized structures which do not support any free charge:



- (a) Sphere of radius R with polarization $\mathbf{P} = (P_0 r/R) \mathbf{i}_r$.
- (b) Permanently polarized slab $P_0 \mathbf{i}_x$ of thickness b placed between parallel plate electrodes in free space at potential difference V_0 .

24. Parallel plate electrodes enclose the series combination of an Ohmic conductor of thickness a with conductivity σ and a superconductor that only has free electrons with plasma

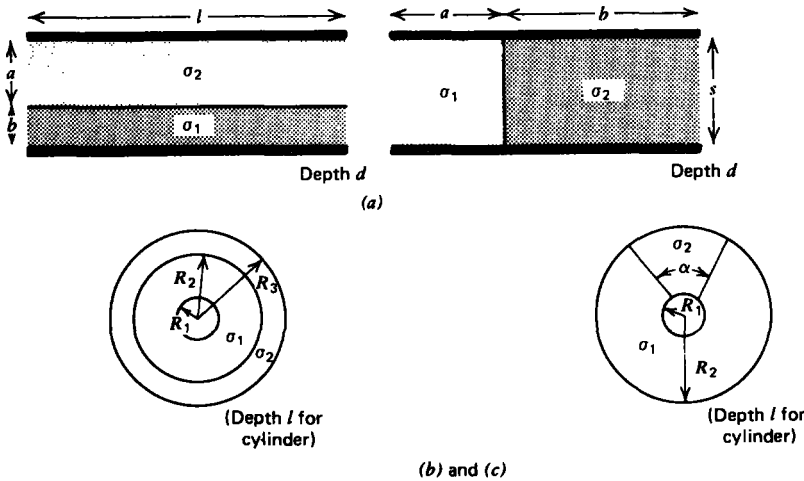


frequency ω_{pe} . What is the time dependence of the terminal current, the electric field in each region, and the surface charge at the interface separating the two conductors for each of the following terminal constraints:

- (a) A step voltage V_0 is applied at $t = 0$. For what values of ω_{pe} are the fields critically damped?
- (b) A sinusoidal voltage $v(t) = V_0 \cos \omega t$ has been applied for a long time.

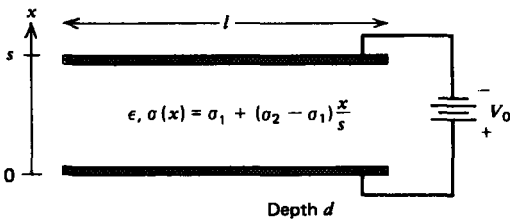
Section 3-4

25. Find the series and parallel resistance between two materials with conductivities σ_1 and σ_2 for each of the following electrode geometries:



- (a) Parallel plates.
- (b) Coaxial cylinders.
- (c) Concentric spheres.

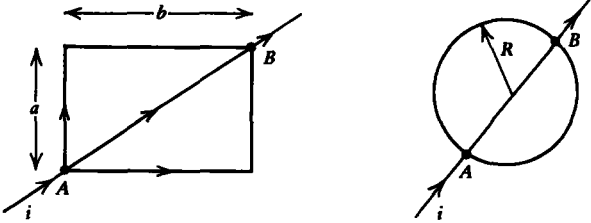
26. A pair of parallel plate electrodes at voltage difference V_0 enclose an Ohmic material whose conductivity varies linearly from σ_1 at the lower electrode to σ_2 at the upper electrode. The permittivity ϵ of the material is a constant.



- (a) Find the fields and the resistance.
- (b) What are the volume and surface charge distributions?

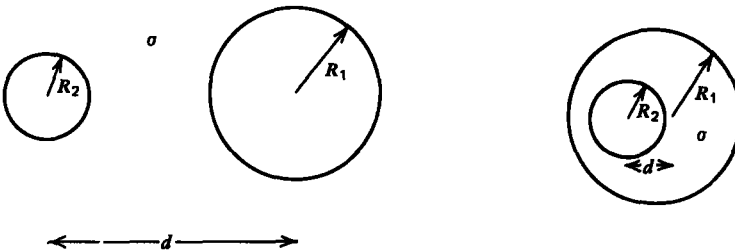
(c) What is the total volume charge in the system and how is it related to the surface charge on the electrodes?

27. A wire of Ohmic conductivity σ and cross sectional area A is twisted into the various shapes shown. What is the resistance R between the points A and B for each of the configurations?



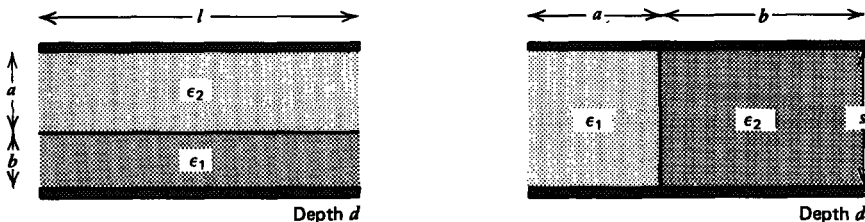
Section 3-5

28. Two conducting cylinders of length l and differing radii R_1 and R_2 within an Ohmic medium with conductivity σ have their centers a distance d apart. What is the resistance between cylinders when they are adjacent and when the smaller one is inside the larger one? (Hint: See Section 2-6-4c.)

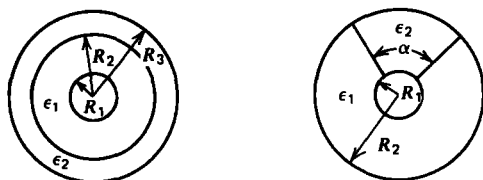


29. Find the series and parallel capacitance for each of the following geometries:

- (a) Parallel plate.
- (b) Coaxial cylinders.
- (c) Concentric spheres.

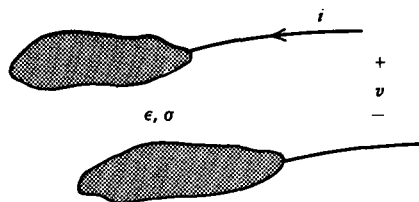


(a)



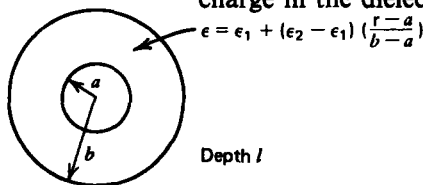
(Depth l for cylinders)
(b), (c)

30. Two arbitrarily shaped electrodes are placed within a medium of constant permittivity ϵ and Ohmic conductivity σ . When a dc voltage V is applied across the system, a current I flows.



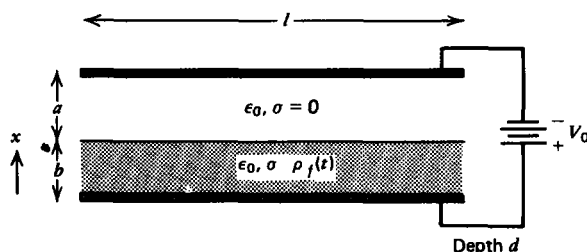
- (a) What is the current $i(t)$ when a sinusoidal voltage $\text{Re}(V_0 e^{j\omega t})$ is applied?
- (b) What is the equivalent circuit of the system?

31. Concentric cylindrical electrodes of length l with respective radii a and b enclose an Ohmic material whose permittivity varies linearly with radius from ϵ_1 at the inner cylinder to ϵ_2 at the outer. What is the capacitance? There is no volume charge in the dielectric.



Section 3.6

32. A lossy material with the permittivity ϵ_0 of free space and conductivity σ partially fills the region between parallel plate electrodes at constant potential difference V_0 and is initially



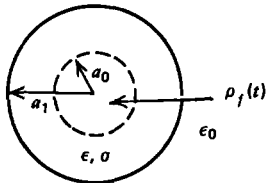
uniformly charged with density ρ_0 at $t = 0$ with zero surface charge at $x = b$. What is the time dependence of the following:

(a) the electric field in each region? (Hint: See Section 3-3-5.)

(b) the surface charge at $x = b$?

(c) the force on the conducting material?

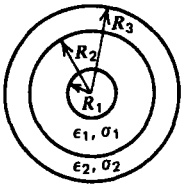
33. An infinitely long cylinder of radius a_1 , permittivity ϵ , and conductivity σ is nonuniformly charged at $t = 0$:



$$\rho_f(t=0) = \begin{cases} \rho_0 r/a_0, & 0 < r < a_0 \\ 0, & r > a_0 \end{cases}$$

What is the time dependence of the electric field everywhere and the surface charge at $r = a_1$?

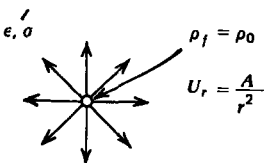
34. Concentric cylindrical electrodes enclose two different media in series. Find the electric field, current density, and surface charges everywhere for each of the following conditions:



Depth l

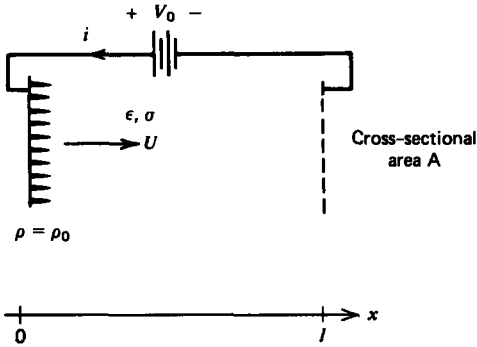
- at $t = 0_+$ right after a step voltage V_0 is applied to the initially unexcited system;
- at $t = \infty$ when the fields have reached their dc steady-state values;
- during the in-between transient interval. (What is the time constant τ ?);
- a sinusoidal voltage $V_0 \cos \omega t$ is applied and has been on a long time;
- what is the equivalent circuit for this system?

35. A fluid flow emanates radially from a point outlet with velocity distribution $U_r = A/r^2$. The fluid has Ohmic conductivity σ and permittivity ϵ . An external source maintains the



charge density ρ_0 at $r = 0$. What are the steady-state charge and electric field distributions throughout space?

36. Charge maintained at constant density ρ_0 at $x = 0$ is carried away by a conducting fluid travelling at constant velocity $U\mathbf{i}_x$ and is collected at $x = l$.



- What are the field and charge distributions within the fluid if the electrodes are at potential difference V_0 ?
- What is the force on the fluid?
- Repeat (a) and (b) if the voltage source is replaced by a load resistor R_L .

37. A dc voltage has been applied a long time to an open circuited resistive-capacitive structure so that the voltage and current have their steady-state distributions as given by (44). Find the resulting discharging transients for voltage and current if at $t = 0$ the terminals at $z = 0$ are suddenly:

- open circuited. **Hint:**

$$\int_0^l \sinh a(z-l) \sin\left(\frac{m\pi z}{l}\right) dz = -\frac{m\pi}{l} \frac{\sinh al}{[a^2 + (m\pi/l)^2]}$$

- Short circuited. **Hint:**

$$\int_0^l \cosh a(z-l) \sin\left(\frac{(2n+1)\pi z}{2l}\right) dz = \frac{(2n+1)\pi \cosh al}{2l \left[a^2 + \left[\frac{(2n+1)\pi}{2l} \right]^2 \right]}$$

38. At $t = 0$ a distributed resistive line as described in Section 3-6-4 has a step dc voltage V_0 applied at $z = 0$. The other end at $z = l$ is short circuited.

- What are the steady-state voltage and current distributions?
- What is the time dependence of the voltage and current during the transient interval? **Hint:**

$$\int_0^l \sinh a(z-l) \sin\left(\frac{m\pi z}{l}\right) dz = -\frac{m\pi \sinh al}{l \left[a^2 + (m\pi/l)^2 \right]}$$

39. A distributed resistive line is excited at $z=0$ with a sinusoidal voltage source $v(t) = V_0 \cos \omega t$ that has been on for a long time. The other end at $z=l$ is either open or short circuited.

(a) Using complex phasor notation of the form

$$v(z, t) = \text{Re} (\hat{v}(z)e^{j\omega t})$$

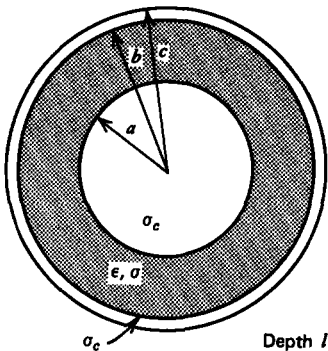
find the sinusoidal steady-state voltage and current distributions for each termination.

(b) What are the complex natural frequencies of the system?

(c) How much time average power is delivered by the source?

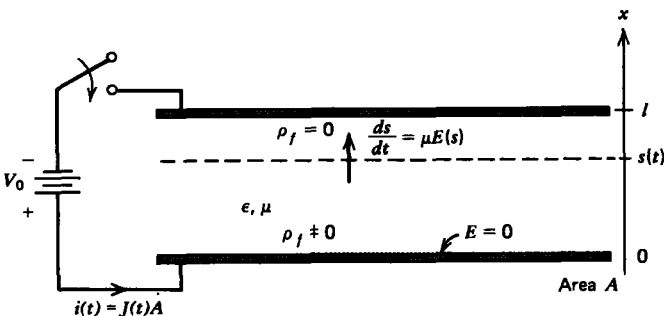
40. A lossy dielectric with permittivity ϵ and Ohmic conductivity σ is placed between coaxial cylindrical electrodes with large Ohmic conductivity σ_c and length l .

What is the series resistance per unit length $2R$ of the electrodes, and the capacitance C and conductance G per unit length of the dielectric?



Section 3.7

41. Two parallel plate electrodes of spacing l enclosing a dielectric with permittivity ϵ are stressed by a step voltage at $t=0$. Positive charge is then injected at $t=0$ from the lower electrode with mobility μ and travels towards the opposite electrode.



(a) Using the charge conservation equation of Section 3-2-1, show that the governing equation is

$$\frac{\partial E}{\partial t} + \mu E \frac{\partial E}{\partial x} = \frac{J(t)}{\epsilon}$$

where $J(t)$ is the current per unit electrode area through the terminal wires. This current does not depend on x .

(b) By integrating (a) between the electrodes, relate the current $J(t)$ solely to the voltage and the electric field at the two electrodes.

(c) For space charge limited conditions ($E(x=0)=0$), find the time dependence of the electric field at the other electrode $E(x=l, t)$ before the charge front reaches it. (**Hint:** With constant voltage, $J(t)$ from (b) only depends on $E(x=l, t)$. Using (a) at $x=l$ with no charge, $\partial E/\partial x = 0$, we have a single differential equation in $E(x=l, t)$.)

(d) What is the electric field acting on the charge front? (**Hint:** There is no charge ahead of the front.)

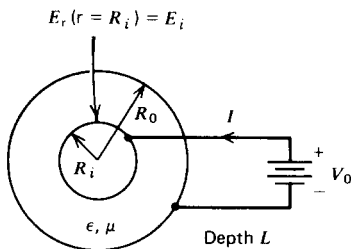
(e) What is the position of the front $s(t)$ as a function of time?

(f) At what time does the front reach the other electrode?

(g) What are the steady-state distribution of potential, electric field, and charge density? What is the steady-state current density $J(t \rightarrow \infty)$?

(h) Repeat (g) for nonspace charge limited conditions when the emitter electric field $E(x=0) = E_0$ is nonzero.

42. In a coaxial cylindrical geometry of length L , the inner electrode at $r = R_i$ is a source of positive ions with mobility μ in the dielectric medium. The inner cylinder is at a dc voltage V_0 with respect to the outer cylinder.



(a) The electric field at the emitter electrode is given as $E_r(r = R_i) = E_i$. If a current I flows, what are the steady-state electric field and space charge distributions?

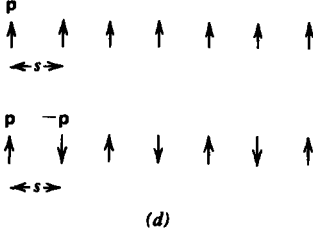
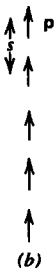
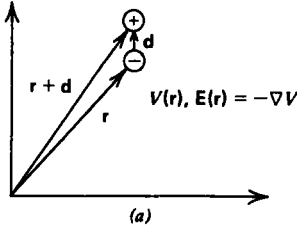
(b) What is the dc current I in terms of the voltage under space charge limited conditions ($E_i = 0$)? **Hint:**

$$\int \frac{[r^2 - R_i^2]^{1/2}}{r} dr = [r^2 - R_i^2]^{1/2} - R_i \cos^{-1}\left(\frac{R_i}{r}\right)$$

- (c) For what value of E_i is the electric field constant between electrodes? What is the resulting current?
- (d) Repeat (a)–(b) for concentric spherical electrodes.

Section 3.8

43. (a) How much work does it take to bring a point dipole from infinity to a position where the electric field is \mathbf{E} ?



- (b) A crystal consists of an infinitely long string of dipoles a constant distance s apart. What is the binding energy of the crystal? (**Hint:** $\sum_{n=1}^{\infty} 1/n^3 \approx 1.2$.)
- (c) Repeat (b) if the dipole moments alternate in sign. (**Hint:** $\sum_{n=1}^{\infty} (-1)^n/n^3 \approx -0.90$.)
- (d) Repeat (b) and (c) if the dipole moments are perpendicular to the line of dipoles for identical or alternating polarity dipoles.

44. What is the energy stored in the field of a point dipole with moment \mathbf{p} outside an encircling concentric sphere with molecular radius R ? **Hint:**

$$\int \cos^2 \theta \sin \theta d\theta = -\frac{\cos^3 \theta}{3}$$

$$\int \sin^3 \theta d\theta = -\frac{1}{3} \cos \theta (\sin^2 \theta + 2)$$

45. A spherical droplet of radius R carrying a total charge Q on its surface is broken up into N identical smaller droplets.
- (a) What is the radius of each droplet and how much charge does it carry?
 - (b) Assuming the droplets are very far apart and do not interact, how much electrostatic energy is stored?

(c) Because of their surface tension the droplets also have a constant surface energy per unit area w_s . What is the total energy (electrostatic plus surface) in the system?

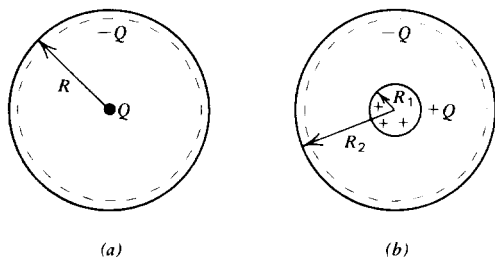
(d) How much work was required to form the droplets and to separate them to infinite spacing.

(e) What value of N minimizes this work? Evaluate for a water droplet with original radius of 1 mm and charge of 10^{-6} coul. (For water $w_s \approx 0.072$ joule/m².)

46. Two coaxial cylinders of radii a and b carry uniformly distributed charge either on their surfaces or throughout the volume. Find the energy stored per unit length in the z direction for each of the following charge distributions that have a total charge of zero:

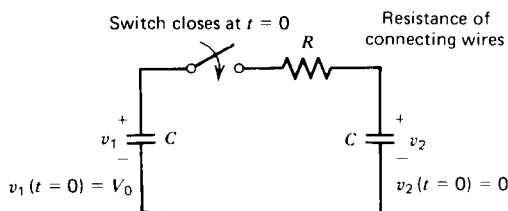
- (a) Surface charge on each cylinder with $\sigma_a 2\pi a = -\sigma_b 2\pi b$.
- (b) Inner cylinder with volume charge ρ_a and outer cylinder with surface charge σ_b where $\sigma_b 2\pi b = -\rho_a \pi a^2$.
- (c) Inner cylinder with volume charge ρ_a with the region between cylinders having volume charge ρ_b where $\rho_a \pi a^2 = -\rho_b \pi (b^2 - a^2)$.

47. Find the binding energy in the following atomic models:



- (a) A point charge Q surrounded by a uniformly distributed surface charge $-Q$ of radius R .
- (b) A uniformly distributed volume charge Q within a sphere of radius R_1 surrounded on the outside by a uniformly distributed surface charge $-Q$ at radius R_2 .

48. A capacitor C is charged to a voltage V_0 . At $t = 0$ another initially uncharged capacitor of equal capacitance C is



connected across the charged capacitor through some lossy wires having an Ohmic conductivity σ , cross-sectional area A , and total length l .

- What is the initial energy stored in the system?
- What is the circuit current i and voltages v_1 and v_2 across each capacitor as a function of time?
- What is the total energy stored in the system in the dc steady state and how does it compare with (a)?
- How much energy has been dissipated in the wire resistance and how does it compare with (a)?
- How do the answers of (b)–(d) change if the system is lossless so that $\sigma = \infty$? How is the power dissipated?
- If the wires are superconducting Section 3-2-5d showed that the current density is related to the electric field as

$$\frac{\partial J}{\partial t} = \omega_p^2 \epsilon E$$

where the plasma frequency ω_p is a constant. What is the equivalent circuit of the system?

- What is the time dependence of the current now?
- How much energy is stored in each element as a function of time?
- At any time t what is the total circuit energy and how does it compare with (a)?

Section 3.9

49. A permanently polarized dipole with moment \mathbf{p} is at an angle θ to a uniform electric field \mathbf{E} .

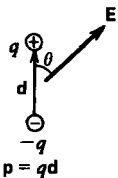
- What is the torque T on the dipole?
- How much incremental work dW is necessary to turn the dipole by a small angle $d\theta$? What is the total work required to move the dipole from $\theta = 0$ to any value of θ ? (**Hint:** $dW = T d\theta$.)

(c) In general, thermal agitation causes the dipoles to be distributed over all angles of θ . Boltzmann statistics tell us that the number density of dipoles having energy W are

$$n = n_0 e^{-W/kT}$$

where n_0 is a constant. If the total number of dipoles within a sphere of radius R is N , what is n_0 ? (**Hint:** Let $u = (pE/kT) \cos \theta$.)

(d) Consider a shell of dipoles within the range of θ to $\theta + d\theta$. What is the magnitude and direction of the net polarization due to this shell?

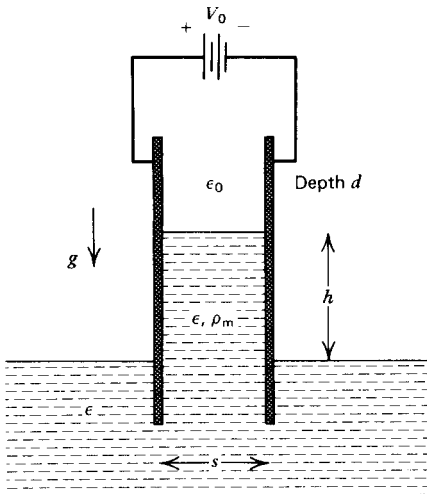


(e) What is the total polarization integrated over θ ? This is known as the Langevin equation. (**Hint:** $\int ue^u du = (u - 1)e^u$.)

(f) Even with a large field of $E \approx 10^6$ v/m with a dipole composed of one proton and electron a distance of 10 \AA (10^{-9} m) apart, show that at room temperature the quantity (pE/kT) is much less than unity and expand the results of (e). (**Hint:** It will be necessary to expand (e) up to third order in (pE/kT) .)

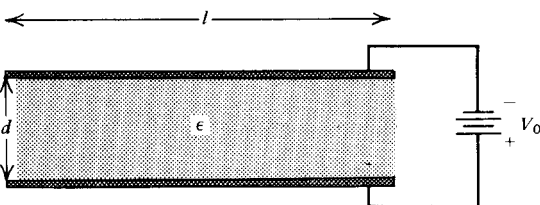
(g) In this limit what is the orientational polarizability?

50. A pair of parallel plate electrodes a distance s apart at a voltage difference V_0 is dipped into a dielectric fluid of permittivity ϵ . The fluid has a mass density ρ_m and gravity acts downward. How high does the liquid rise between the plates?



51. Parallel plate electrodes at voltage difference V_0 enclose an elastic dielectric with permittivity ϵ . The electric force of attraction between the electrodes is balanced by the elastic force of the dielectric.

(a) When the electrode spacing is d what is the free surface charge density on the upper electrode?



(b) What is the electric force per unit area that the electrode exerts on the dielectric interface?

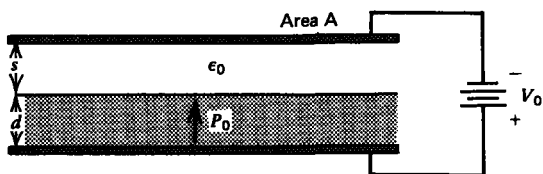
(c) The elastic restoring force per unit area is given by the relation

$$F_A = Y \ln \frac{d}{d_0}$$

where Y is the modulus of elasticity and d_0 is the unstressed ($V_0 = 0$) thickness of the dielectric. Write a transcendental expression for the equilibrium thickness of the dielectric.

(d) What is the minimum equilibrium dielectric thickness and at what voltage does it occur? If a larger voltage is applied there is no equilibrium and the dielectric fractures as the electric stress overcomes the elastic restoring force. This is called the theory of electromechanical breakdown. [See K. H. Stark and C. G. Garton, *Electric Strength of Irradiated Polythene*, *Nature* 176 (1955) 1225–26.]

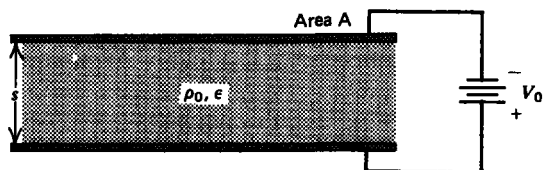
52. An electret with permanent polarization $P_0 \mathbf{i}$, and thickness d partially fills a free space capacitor. There is no surface charge on the electret free space interface.



(a) What are the electric fields in each region?

(b) What is the force on the upper electrode?

53. A uniform distribution of free charge with density ρ_0 is between parallel plate electrodes at potential difference V_0 .



(a) What is the energy stored in the system?

(b) Compare the capacitance to that when $\rho_0 = 0$.

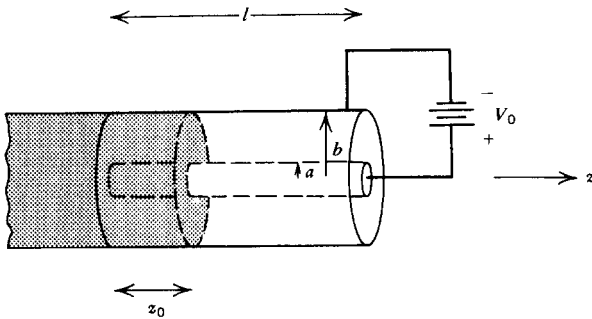
(c) What is the total force on each electrode and on the volume charge distribution?

(d) What is the total force on the system?

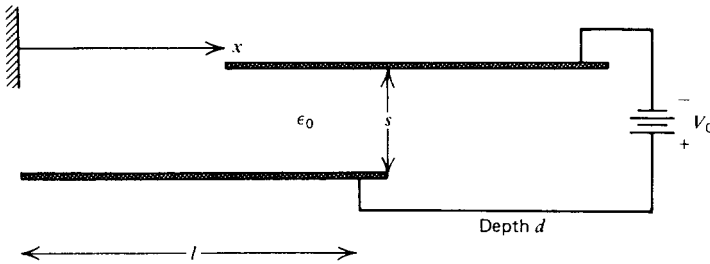
54. Coaxial cylindrical electrodes at voltage difference V_0 are partially filled with a polarized material. Find the force on this

material if it is

- permanently polarized as $P_0 \hat{i}_r$;
- linearly polarized with permittivity ϵ .



55. The upper electrode of a pair at constant potential difference V_0 is free to slide in the x direction. What is the x component of the force on the upper electrode?



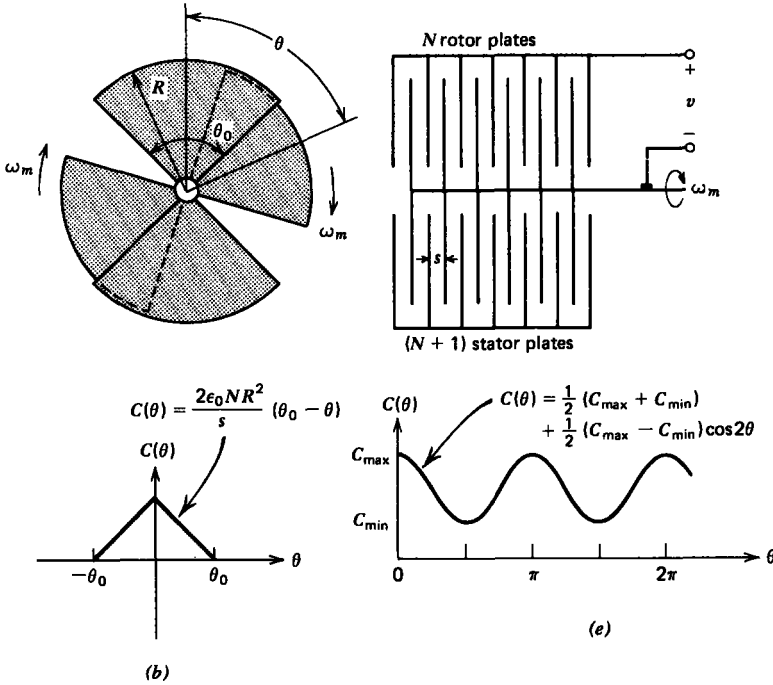
56. A capacitor has a moveable part that can rotate through the angle θ so that the capacitance $C(\theta)$ depends on θ .

- What is the torque on the moveable part?
- An electrostatic voltmeter consists of $N+1$ fixed pie-shaped electrodes at the same potential interspersed with N plates mounted on a shaft that is free to rotate for $-\theta_0 < \theta < \theta_0$. What is the capacitance as a function of θ ?
- A voltage v is applied. What is the electric torque on the shaft?
- A torsional spring exerts a restoring torque on the shaft

$$T_s = -K(\theta - \theta_s)$$

where K is the spring constant and θ_s is the equilibrium position of the shaft at zero voltage. What is the equilibrium position of the shaft when the voltage v is applied? If a sinusoidal voltage is applied, what is the time average angular deflection $\langle \theta \rangle$?

- The torsional spring is removed so that the shaft is free to continuously rotate. Fringing field effects cause the



capacitance to vary smoothly between minimum and maximum values of a dc value plus a single sinusoidal spatial term

$$C(\theta) = \frac{1}{2}[C_{\max} + C_{\min}] + \frac{1}{2}[C_{\max} - C_{\min}] \cos 2\theta$$

A sinusoidal voltage $V_0 \cos \omega t$ is applied. What is the instantaneous torque on the shaft?

(f) If the shaft is rotating at constant angular speed ω_m so that

$$\theta = \omega_m t + \delta$$

where δ is the angle of the shaft at $t = 0$, under what conditions is the torque in (e) a constant? **Hint:**

$$\begin{aligned} \sin 2\theta \cos^2 \omega t &= \frac{1}{2} \sin 2\theta (1 + \cos 2\omega t) \\ &= \frac{1}{2} \sin' 2\theta + \frac{1}{4} [\sin (2(\omega t + \theta)) - \sin (2(\omega t - \theta))] \end{aligned}$$

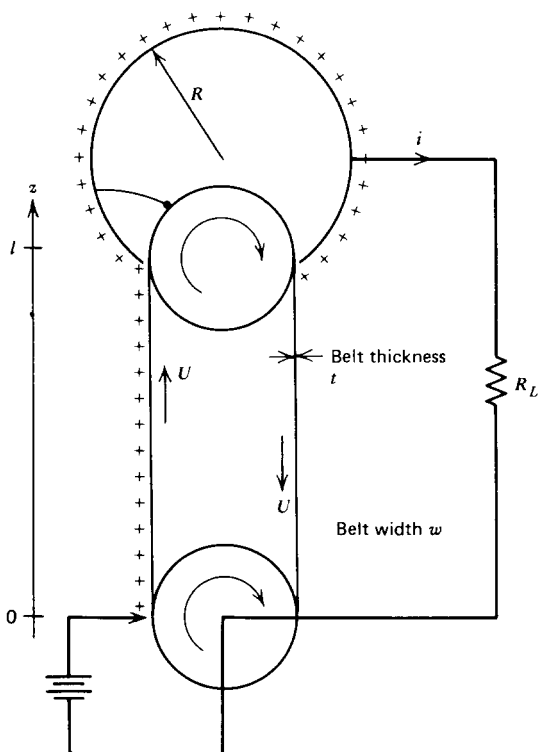
(g) A time average torque T_0 is required of the shaft. What is the torque angle δ ?

(h) What is the maximum torque that can be delivered? This is called the pull-out torque. At what angle δ does this occur?

Section 3-10

57. The belt of a Van de Graaff generator has width w and moves with speed U carrying a surface charge σ_f up to the spherical dome of radius R .

- (a) What is the time dependence of the dome voltage?
 (b) Assuming that the electric potential varies linearly between the charging point and the dome, how much power as a function of time is required for the motor to rotate the belt?

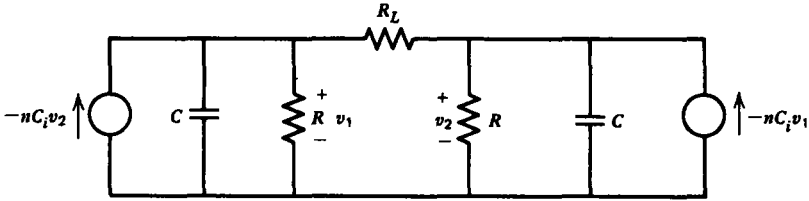


58. A Van de Graaff generator has a lossy belt with Ohmic conductivity σ traveling at constant speed U . The charging point at $z = 0$ maintains a constant volume charge density ρ_0 on the belt at $z = 0$. The dome is loaded by a resistor R_L to ground.

- (a) Assuming only one-dimensional variations with z , what are the steady-state volume charge, electric field, and current density distributions on the belt?
 (b) What is the steady-state dome voltage?

59. A pair of coupled electrostatic induction machines have their inducer electrodes connected through a load resistor R_L . In addition, each electrode has a leakage resistance R to ground.

- (a) For what values of n , the number of conductors per second passing the collector, will the machine self-excite?



(b) If $n = 10$, $C_i = 2$ pf, and $C = 10$ pf with $R_L = R$, what is the minimum value of R for self-excitation?

(c) If we have three such coupled machines, what is the condition for self-excitation and what are the oscillation frequencies if $R_L = \infty$?

(d) Repeat (c) for N such coupled machines with $R_L = \infty$. The last machine is connected to the first.