chapter 1

review of vector analysis

Electromagnetic field theory is the study of forces between charged particles resulting in energy conversion or signal transmission and reception. These forces vary in magnitude and direction with time and throughout space so that the theory is a heavy user of vector, differential, and integral calculus. This chapter presents a brief review that highlights the essential mathematical tools needed throughout the text. We isolate the mathematical details here so that in later chapters most of our attention can be devoted to the applications of the mathematics rather than to its development. Additional mathematical material will be presented as needed throughout the text.

1-1 COORDINATE SYSTEMS

A coordinate system is a way of uniquely specifying the location of any position in space with respect to a reference origin. Any point is defined by the intersection of three mutually perpendicular surfaces. The coordinate axes are then defined by the normals to these surfaces at the point. Of course the solution to any problem is always independent of the choice of coordinate system used, but by taking advantage of symmetry, computation can often be simplified by proper choice of coordinate description. In this text we only use the familiar rectangular (Cartesian), circular cylindrical, and spherical coordinate systems.

1-1-1 Rectangular (Cartesian) Coordinates

The most common and often preferred coordinate system is defined by the intersection of three mutually perpendicular planes as shown in Figure 1-1*a*. Lines parallel to the lines of intersection between planes define the coordinate axes (x, y, z), where the x axis lies perpendicular to the plane of constant x, the y axis is perpendicular to the plane of constant y, and the z axis is perpendicular to the plane of constant z. Once an origin is selected with coordinate $(0, 0, 0)$, any other point in the plane is found by specifying its x-directed, y-directed, and z-directed distances from this origin as shown for the coordinate points located in Figure 1-1*b*.

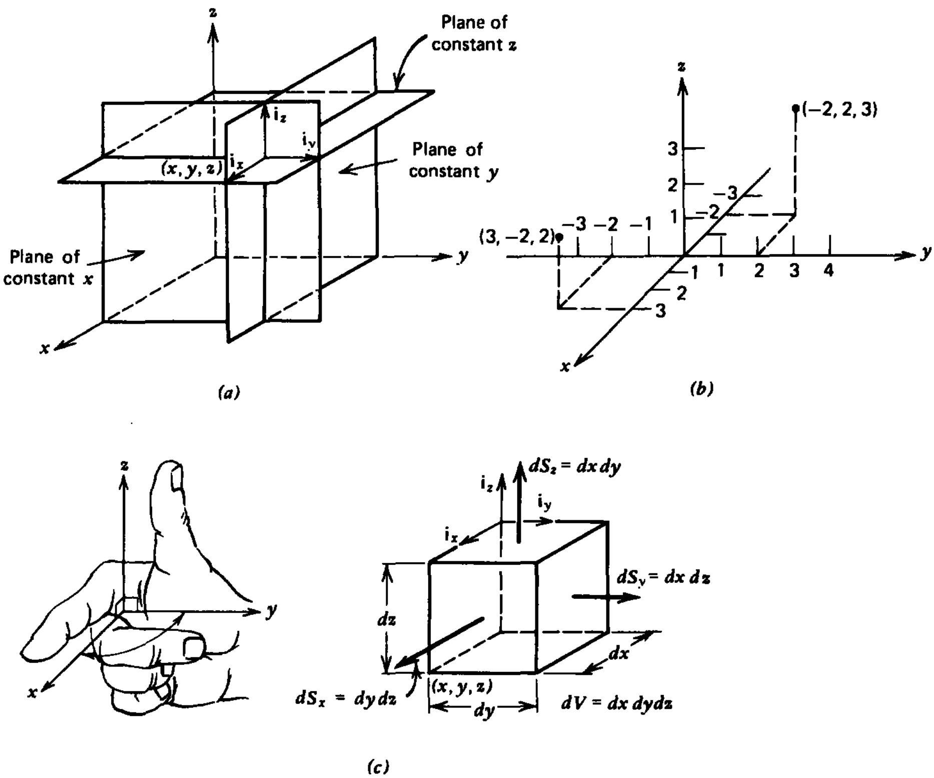

Figure 1-1 Cartesian coordinate system. (*a*) Intersection of three mutually perpendicular planes defines the Cartesian coordinates (x, y, z). (*b*) A point is located in space by specifying its x-, y- and z-directed distances from the origin. (*c*) Differential volume and surface area elements.

By convention, a right-handed coordinate system is always used whereby one curls the fingers of his or her right hand in the direction from x to y so that the forefinger is in the x direction and the middle finger is in the y direction. The thumb then points in the z direction. This convention is necessary to remove directional ambiguities in theorems to be derived later.

Coordinate directions are represented by unit vectors $\mathbf{i}_x$, $\mathbf{i}_y$ and $\mathbf{i}_z$, each of which has a unit length and points in the direction along one of the coordinate axes. Rectangular coordinates are often the simplest to use because the unit vectors always point in the same direction and do not change direction from point to point.

A rectangular differential volume is formed when one moves from a point (x, y, z) by an incremental distance dx, dy, and dz in each of the three coordinate directions as shown in

Figure 1-1c. To distinguish surface elements we subscript the area element of each face with the coordinate perpendicular to the surface.

1-1-2 Circular Cylindrical Coordinates

The cylindrical coordinate system is convenient to use when there is a line of symmetry that is defined as the z axis. As shown in Figure 1-2*a*, any point in space is defined by the intersection of the three perpendicular surfaces of a circular cylinder of radius r, a plane at constant z, and a plane at constant angle ϕ from the x axis.

The unit vectors $\mathbf{i}_r$, $\mathbf{i}_\phi$ and $\mathbf{i}_z$ are perpendicular to each of these surfaces. The direction of $\mathbf{i}_z$ is independent of position, but unlike the rectangular unit vectors the direction of $\mathbf{i}_r$ and $\mathbf{i}_\phi$ change with the angle ϕ as illustrated in Figure 1-2*b*. For instance, when $\phi=0$ then $\mathbf{i}_r=\mathbf{i}_x$ and $\mathbf{i}_\phi=\mathbf{i}_y$, while if $\phi=\pi/2$, then $\mathbf{i}_r=\mathbf{i}_y$ and $\mathbf{i}_\phi=-\mathbf{i}_x$.

By convention, the triplet (r, ϕ, z) must form a right-handed coordinate system so that curling the fingers of the right hand from $\mathbf{i}_r$ to $\mathbf{i}_\phi$ puts the thumb in the z direction.

A section of differential size cylindrical volume, shown in Figure 1-2*c*, is formed when one moves from a point at coordinate (r, ϕ, z) by an incremental distance dr, r $d\phi$, and dz in each of the three coordinate directions. The differential volume and surface areas now depend on the coordinate r as summarized in Table 1-1.

Table 1-1 Differential lengths, surface area, and volume elements for each geometry. The surface element is subscripted by the coordinate perpendicular to the surface

CARTESIAN	CYLINDRICAL	SPHERICAL
$\mathbf{dl}=dx\,\mathbf{i}_x+dy\,\mathbf{i}_y+dz\,\mathbf{i}_z$	$\mathbf{dl}=dr\,\mathbf{i}_r+r\,d\phi\,\mathbf{i}_\phi+dz\,\mathbf{i}_z$	$\mathbf{dl}=dr\,\mathbf{i}_r+r\,d\theta\,\mathbf{i}_\theta$ $+r\sin\theta\,d\phi\,\mathbf{i}_\phi$
$dS_x=dy\,dz$	$dS_r=r\,d\phi\,dz$	$dS_r=r^2\sin\theta\,d\theta\,d\phi$
$dS_y=dx\,dz$	$dS_\phi=dr\,dz$	$dS_\theta=r\sin\theta\,dr\,d\phi$
$dS_z=dx\,dy$	$dS_z=r\,dr\,d\phi$	$dS_\phi=r\,dr\,d\theta$
$dV=dx\,dy\,dz$	$dV=r\,dr\,d\phi\,dz$	$dV=r^2\sin\theta\,dr\,d\theta\,d\phi$

1-1-3 Spherical Coordinates

A spherical coordinate system is useful when there is a point of symmetry that is taken as the origin. In Figure 1-3*a* we see that the spherical coordinate (r, θ, ϕ) is obtained by the intersection of a sphere with radius r, a plane at constant

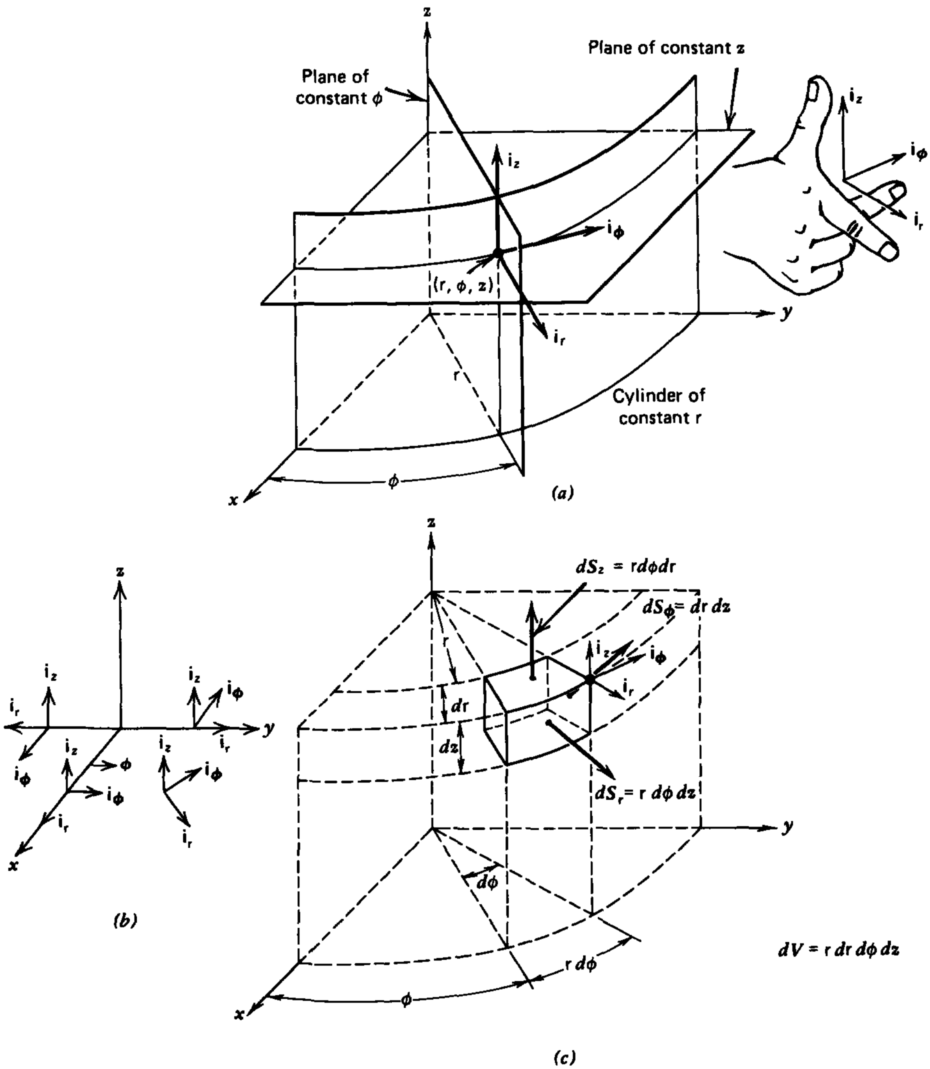

Figure 1-2 Circular cylindrical coordinate system. (*a*) Intersection of planes of constant z and ϕ with a cylinder of constant radius r defines the coordinates (r, ϕ, z). (*b*) The direction of the unit vectors $\mathbf{i}_r$ and $\mathbf{i}_\phi$ vary with the angle ϕ. (*c*) Differential volume and surface area elements.

angle ϕ from the x axis as defined for the cylindrical coordinate system, and a cone at angle θ from the z axis. The unit vectors $\mathbf{i}_r$, $\mathbf{i}_\theta$ and $\mathbf{i}_\phi$ are perpendicular to each of these surfaces and change direction from point to point. The triplet (r, θ, ϕ) must form a right-handed set of coordinates.

The differential-size spherical volume element formed by considering incremental displacements dr, $r\,d\theta$, $r \sin\theta\, d\phi$

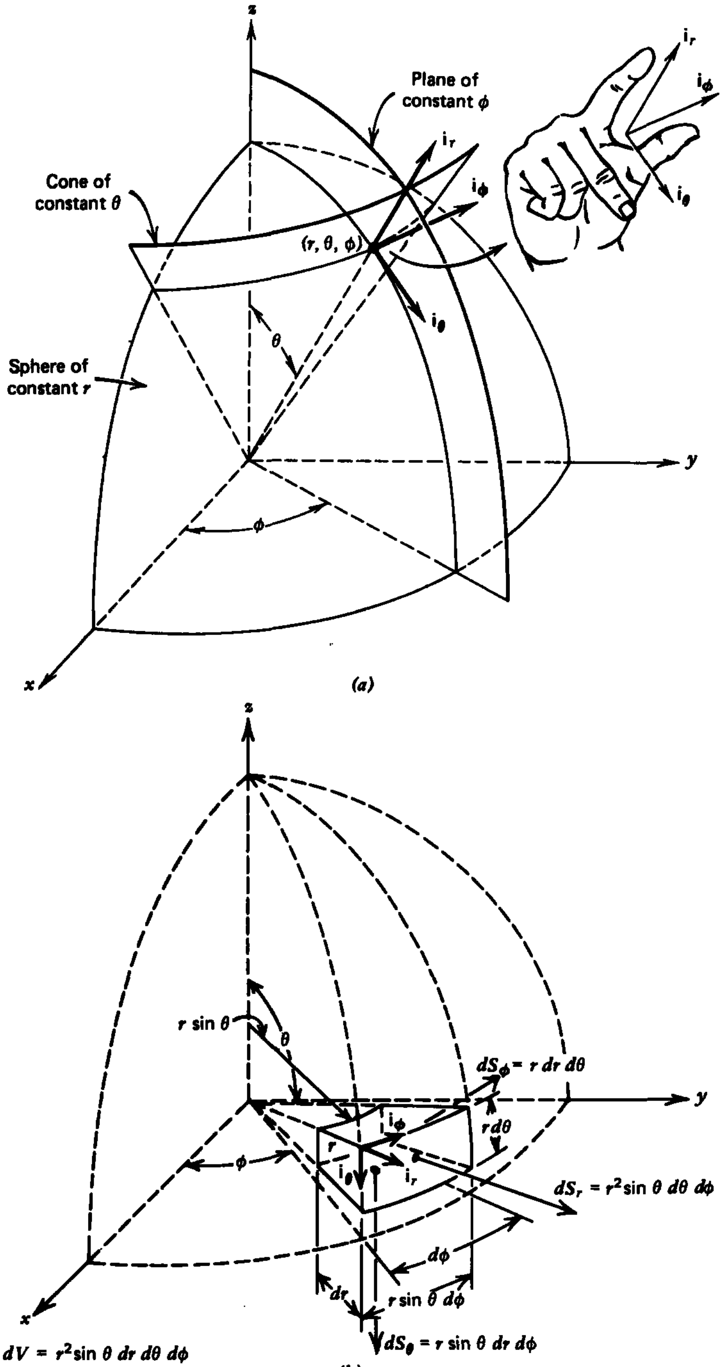

Figure 1-3 Spherical coordinate system. (*a*) Intersection of plane of constant angle ϕ with cone of constant angle θ and sphere of constant radius r defines the coordinates (r, θ, ϕ). (*b*) Differential volume and surface area elements.

Table 1-2 Geometric relations between coordinates and unit vectors for Cartesian, cylindrical, and spherical coordinate systems*

CARTESIAN		CYLINDRICAL		SPHERICAL
x	=	$\mathrm{r}\cos\phi$	=	$r\sin\theta\cos\phi$
y	=	$\mathrm{r}\sin\phi$	=	$r\sin\theta\sin\phi$
z	=	z	=	$r\cos\theta$
$\mathbf{i}_x$	=	$\cos\phi\,\mathbf{i}_r-\sin\phi\,\mathbf{i}_\phi$	=	$\sin\theta\cos\phi\,\mathbf{i}_r+\cos\theta\cos\phi\,\mathbf{i}_\theta-\sin\phi\,\mathbf{i}_\phi$
$\mathbf{i}_y$	=	$\sin\phi\,\mathbf{i}_r+\cos\phi\,\mathbf{i}_\phi$	=	$\sin\theta\sin\phi\,\mathbf{i}_r+\cos\theta\sin\phi\,\mathbf{i}_\theta+\cos\phi\,\mathbf{i}_\phi$
$\mathbf{i}_z$	=	$\mathbf{i}_z$	=	$\cos\theta\,\mathbf{i}_r-\sin\theta\,\mathbf{i}_\theta$
CYLINDRICAL		CARTESIAN		SPHERICAL
r	=	$\sqrt{x^2+y^2}$	=	$r\sin\theta$
ϕ	=	$\tan^{-1}\frac{y}{x}$	=	ϕ
z	=	z	=	$r\cos\theta$
$\mathbf{i}_r$	=	$\cos\phi\,\mathbf{i}_x+\sin\phi\,\mathbf{i}_y$	=	$\sin\theta\,\mathbf{i}_r+\cos\theta\,\mathbf{i}_\theta$
$\mathbf{i}_\phi$	=	$-\sin\phi\,\mathbf{i}_x+\cos\phi\,\mathbf{i}_y$	=	$\mathbf{i}_\phi$
$\mathbf{i}_z$	=	$\mathbf{i}_z$	=	$\cos\theta\,\mathbf{i}_r-\sin\theta\,\mathbf{i}_\theta$
SPHERICAL		CARTESIAN		CYLINDRICAL
r	=	$\sqrt{x^2+y^2+z^2}$	=	$\sqrt{\mathrm{r}^2+z^2}$
θ	=	$\cos^{-1}\frac{z}{\sqrt{x^2+y^2+z^2}}$	=	$\cos^{-1}\frac{z}{\sqrt{\mathrm{r}^2+z^2}}$
ϕ	=	$\cot^{-1}\frac{x}{y}$	=	ϕ
$\mathbf{i}_r$	=	$\sin\theta\cos\phi\,\mathbf{i}_x+\sin\theta\sin\phi\,\mathbf{i}_y+\cos\theta\,\mathbf{i}_z$	=	$\sin\theta\,\mathbf{i}_r+\cos\theta\,\mathbf{i}_z$
$\mathbf{i}_\theta$	=	$\cos\theta\cos\phi\,\mathbf{i}_x+\cos\theta\sin\phi\,\mathbf{i}_y-\sin\theta\,\mathbf{i}_z$	=	$\cos\theta\,\mathbf{i}_r-\sin\theta\,\mathbf{i}_z$
$\mathbf{i}_\phi$	=	$-\sin\phi\,\mathbf{i}_x+\cos\phi\,\mathbf{i}_y$	=	$\mathbf{i}_\phi$

* *Note that throughout this text a lower case roman* r *is used for the cylindrical radial coordinate while an italicized* r *is used for the spherical radial coordinate.*

from the coordinate (r, θ, ϕ) now depends on the angle θ and the radial position r as shown in Figure 1-3b and summarized in Table 1-1. Table 1-2 summarizes the geometric relations between coordinates and unit vectors for the three coordinate systems considered. Using this table, it is possible to convert coordinate positions and unit vectors from one system to another.

1-2 VECTOR ALGEBRA

1-2-1 Scalars and Vectors

A scalar quantity is a number completely determined by its magnitude, such as temperature, mass, and charge, the last being especially important in our future study. Vectors, such as velocity and force, must also have their direction specified and in this text are printed in boldface type. They are completely described by their components along three coordinate directions as shown for rectangular coordinates in Figure 1-4. A vector is represented by a directed line segment in the direction of the vector with its length proportional to its magnitude. The vector

$$\mathbf{A} = A_x\mathbf{i}_x + A_y\mathbf{i}_y + A_z\mathbf{i}_z \tag{1}$$

in Figure 1-4 has magnitude

$$A = |\mathbf{A}| = [A_x^2 + A_y^2 + A_z^2]^{1/2} \tag{2}$$

Note that each of the components in (1) (A_x, A_y, and A_z) are themselves scalars. The direction of each of the components is given by the unit vectors. We could describe a vector in any of the coordinate systems replacing the subscripts (x, y, z) by (r, ϕ, z) or (r, θ, ϕ); however, for conciseness we often use rectangular coordinates for general discussion.

1-2-2 Multiplication of a Vector by a Scalar

If a vector is multiplied by a positive scalar, its direction remains unchanged but its magnitude is multiplied by the

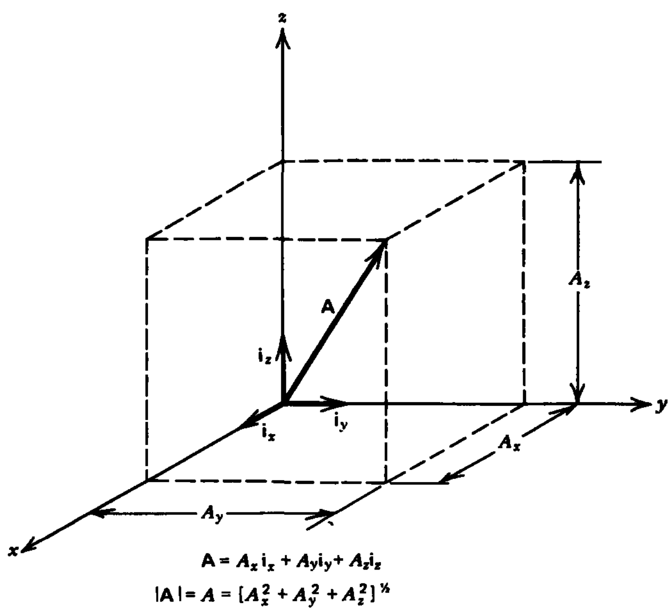

Figure 1-4 A vector is described by its components along the three coordinate directions.

scalar. If the scalar is negative, the direction of the vector is reversed:

$$a\mathbf{A} = aA_x\mathbf{i}_x + aA_y\mathbf{i}_y + aA_z\mathbf{i}_z \tag{3}$$

1-2-3 Addition and Subtraction

The sum of two vectors is obtained by adding their components while their difference is obtained by subtracting their components. If the vector **B**

$$\mathbf{B} = B_x\mathbf{i}_x + B_y\mathbf{i}_y + B_z\mathbf{i}_z \tag{4}$$

is added or subtracted to the vector **A** of (1), the result is a new vector **C**:

$$\mathbf{C} = \mathbf{A} \pm \mathbf{B} = (A_x \pm B_x)\mathbf{i}_x + (A_y \pm B_y)\mathbf{i}_y + (A_z \pm B_z)\mathbf{i}_z \tag{5}$$

Geometrically, the vector sum is obtained from the diagonal of the resulting parallelogram formed from **A** and **B** as shown in Figure 1-5*a*. The difference is found by first

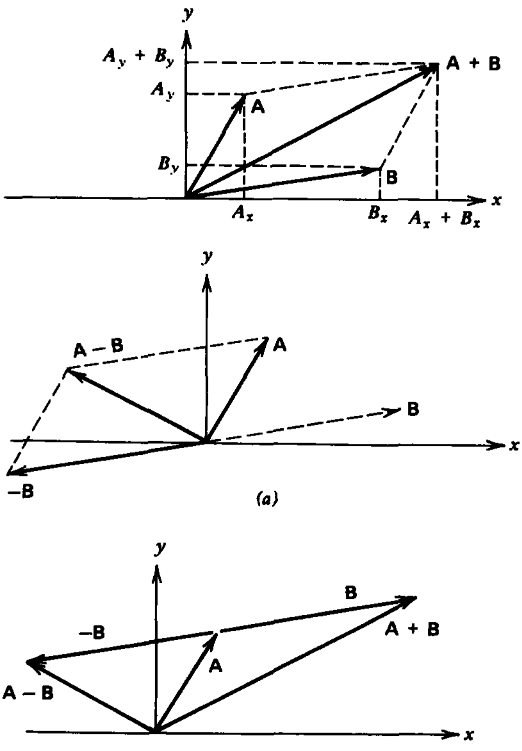

Figure 1-5 The sum and difference of two vectors (*a*) by finding the diagonal of the parallelogram formed by the two vectors, and (*b*) by placing the tail of a vector at the head of the other.

drawing $-\mathbf{B}$ and then finding the diagonal of the parallelogram formed from the sum of **A** and $-\mathbf{B}$. The sum of the two vectors is equivalently found by placing the tail of a vector at the head of the other as in Figure 1-5*b*.

Subtraction is the same as addition of the negative of a vector.

EXAMPLE 1-1 VECTOR ADDITION AND SUBTRACTION

Given the vectors

$$\mathbf{A} = 4\mathbf{i}_x + 4\mathbf{i}_y, \quad \mathbf{B} = \mathbf{i}_x + 8\mathbf{i}_y$$

find the vectors $\mathbf{B} \pm \mathbf{A}$ and their magnitudes. For the geometric solution, see Figure 1-6.

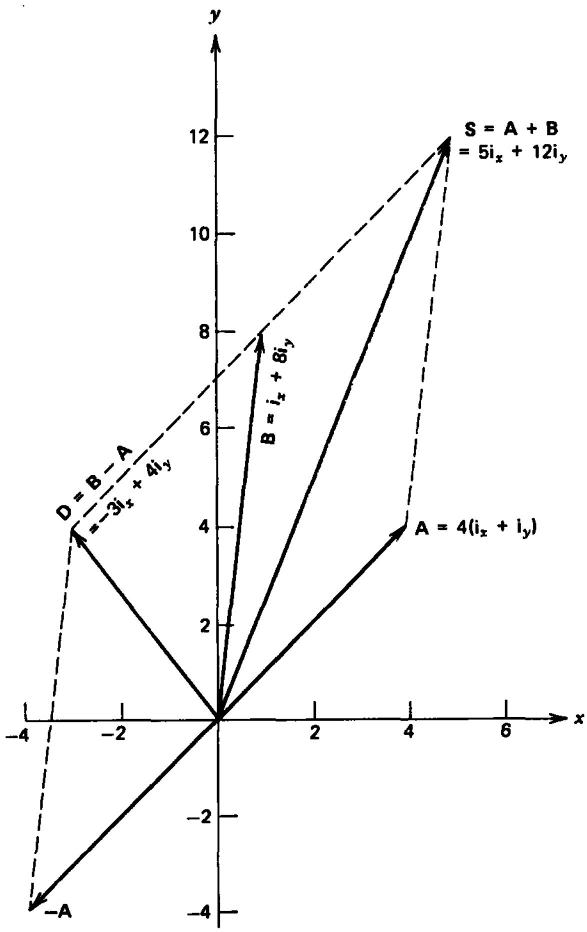

Figure 1-6 The sum and difference of vectors **A** and **B** given in Example 1-1.

SOLUTION

Sum

$$\mathbf{S} = \mathbf{A} + \mathbf{B} = (4+1)\mathbf{i}_x + (4+8)\mathbf{i}_y = 5\mathbf{i}_x + 12\mathbf{i}_y$$

$$S = [5^2 + 12^2]^{1/2} = 13$$

Difference

$$\mathbf{D} = \mathbf{B} - \mathbf{A} = (1-4)\mathbf{i}_x + (8-4)\mathbf{i}_y = -3\mathbf{i}_x + 4\mathbf{i}_y$$

$$D = [(-3)^2 + 4^2]^{1/2} = 5$$

1-2-4 The Dot (Scalar) Product

The dot product between two vectors results in a scalar and is defined as

$$\mathbf{A} \cdot \mathbf{B} = AB \cos\theta \tag{6}$$

where θ is the smaller angle between the two vectors. The term $A \cos\theta$ is the component of the vector **A** in the direction of **B** shown in Figure 1-7. One application of the dot product arises in computing the incremental work dW necessary to move an object a differential vector distance **dl** by a force **F**. Only the component of force in the direction of displacement contributes to the work

$$dW = \mathbf{F} \cdot \mathbf{dl} \tag{7}$$

The dot product has maximum value when the two vectors are colinear ($\theta = 0$) so that the dot product of a vector with itself is just the square of its magnitude. The dot product is zero if the vectors are perpendicular ($\theta = \pi/2$). These properties mean that the dot product between different orthogonal unit vectors at the same point is zero, while the dot

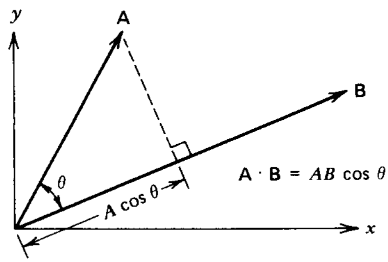

Figure 1-7 The dot product between two vectors.

product between a unit vector and itself is unity

$$\begin{aligned} \mathbf{i}_x \cdot \mathbf{i}_x &= 1, \quad & \mathbf{i}_x \cdot \mathbf{i}_y &= 0 \\ \mathbf{i}_y \cdot \mathbf{i}_y &= 1, \quad & \mathbf{i}_x \cdot \mathbf{i}_z &= 0 \\ \mathbf{i}_z \cdot \mathbf{i}_z &= 1, \quad & \mathbf{i}_y \cdot \mathbf{i}_z &= 0 \end{aligned} \tag{8}$$

Then the dot product can also be written as

$$\begin{aligned} \mathbf{A} \cdot \mathbf{B} &= (A_x \mathbf{i}_x + A_y \mathbf{i}_y + A_z \mathbf{i}_z) \cdot (B_x \mathbf{i}_x + B_y \mathbf{i}_y + B_z \mathbf{i}_z) \\ &= A_x B_x + A_y B_y + A_z B_z \end{aligned} \tag{9}$$

From (6) and (9) we see that the dot product does not depend on the order of the vectors

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A} \tag{10}$$

By equating (6) to (9) we can find the angle between vectors as

$$\cos \theta = \frac{A_x B_x + A_y B_y + A_z B_z}{AB} \tag{11}$$

Similar relations to (8) also hold in cylindrical and spherical coordinates if we replace (x, y, z) by (r, ϕ, z) or (r, θ, ϕ). Then (9) to (11) are also true with these coordinate substitutions.

EXAMPLE 1-2 DOT PRODUCT

Find the angle between the vectors shown in Figure 1-8,

$$\mathbf{A} = \sqrt{3}\, \mathbf{i}_x + \mathbf{i}_y, \quad \mathbf{B} = 2\mathbf{i}_x$$

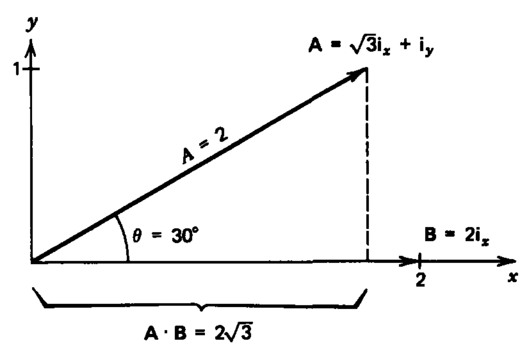

Figure 1-8 The angle between the two vectors **A** and **B** in Example 1-2 can be found using the dot product.

SOLUTION

From (11)

$$\cos\theta = \frac{A_x B_x}{[A_x^2 + A_y^2]^{1/2} B_x} = \frac{\sqrt{3}}{2}$$

$$\theta = \cos^{-1}\frac{\sqrt{3}}{2} = 30^\circ$$

1-2-5 The Cross (Vector) Product

The cross product between two vectors $\mathbf{A}\times\mathbf{B}$ is defined as a vector perpendicular to both **A** and **B**, which is in the direction of the thumb when using the right-hand rule of curling the fingers of the right hand from **A** to **B** as shown in Figure 1-9. The magnitude of the cross product is

$$|\mathbf{A}\times\mathbf{B}| = AB\sin\theta \tag{12}$$

where θ is the enclosed angle between **A** and **B**. Geometrically, (12) gives the area of the parallelogram formed with **A** and **B** as adjacent sides. Interchanging the order of **A** and **B** reverses the sign of the cross product:

$$\mathbf{A}\times\mathbf{B} = -\mathbf{B}\times\mathbf{A} \tag{13}$$

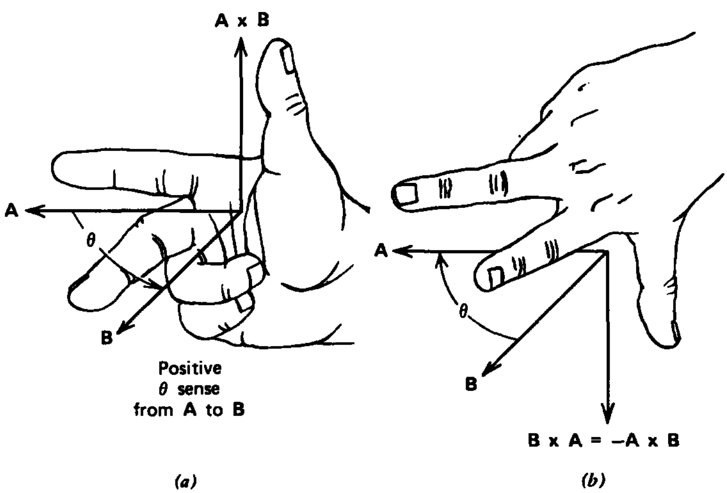

Figure 1-9 (*a*) The cross product between two vectors results in a vector perpendicular to both vectors in the direction given by the right-hand rule. (*b*) Changing the order of vectors in the cross product reverses the direction of the resultant vector.

The cross product is zero for colinear vectors ($\theta = 0$) so that the cross product between a vector and itself is zero and is maximum for perpendicular vectors ($\theta = \pi/2$). For rectangular unit vectors we have

$$\begin{aligned} \mathbf{i}_x \times \mathbf{i}_x &= 0, & \mathbf{i}_x \times \mathbf{i}_y &= \mathbf{i}_z, & \mathbf{i}_y \times \mathbf{i}_x &= -\mathbf{i}_z \\ \mathbf{i}_y \times \mathbf{i}_y &= 0, & \mathbf{i}_y \times \mathbf{i}_z &= \mathbf{i}_x, & \mathbf{i}_z \times \mathbf{i}_y &= -\mathbf{i}_x \\ \mathbf{i}_z \times \mathbf{i}_z &= 0, & \mathbf{i}_z \times \mathbf{i}_x &= \mathbf{i}_y, & \mathbf{i}_x \times \mathbf{i}_z &= -\mathbf{i}_y \end{aligned} \tag{14}$$

These relations allow us to simply define a right-handed coordinate system as one where

$$\mathbf{i}_x \times \mathbf{i}_y = \mathbf{i}_z \tag{15}$$

Similarly, for cylindrical and spherical coordinates, right-handed coordinate systems have

$$\mathbf{i}_r \times \mathbf{i}_\phi = \mathbf{i}_z, \quad \mathbf{i}_r \times \mathbf{i}_\theta = \mathbf{i}_\phi \tag{16}$$

The relations of (14) allow us to write the cross product between **A** and **B** as

$$\begin{aligned} \mathbf{A} \times \mathbf{B} &= (A_x \mathbf{i}_x + A_y \mathbf{i}_y + A_z \mathbf{i}_z) \times (B_x \mathbf{i}_x + B_y \mathbf{i}_y + B_z \mathbf{i}_z) \\ &= \mathbf{i}_x (A_y B_z - A_z B_y) + \mathbf{i}_y (A_z B_x - A_x B_z) + \mathbf{i}_z (A_x B_y - A_y B_x) \end{aligned} \tag{17}$$

which can be compactly expressed as the determinantal expansion

$$\begin{aligned} \mathbf{A} \times \mathbf{B} &= \det \begin{vmatrix} \mathbf{i}_x & \mathbf{i}_y & \mathbf{i}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \\ &= \mathbf{i}_x (A_y B_z - A_z B_y) + \mathbf{i}_y (A_z B_x - A_x B_z) + \mathbf{i}_z (A_x B_y - A_y B_x) \end{aligned} \tag{18}$$

The cyclical and orderly permutation of (x, y, z) allows easy recall of (17) and (18). If we think of xyz as a three-day week where the last day z is followed by the first day x, the days progress as

$$\underline{xyz}\,x\,\underline{yzx}\,y\,\underline{zxy}\,z \cdots \tag{19}$$

where the three possible positive permutations are underlined. Such permutations of xyz in the subscripts of (18) have positive coefficients while the odd permutations, where xyz do not follow sequentially

$$xzy, \; yxz, \; zyx \tag{20}$$

have negative coefficients in the cross product.

In (14)–(20) we used Cartesian coordinates, but the results remain unchanged if we sequentially replace (x, y, z) by the cylindrical coordinates (r, ϕ, z) or the spherical coordinates (r, θ, ϕ).

EXAMPLE 1-3 CROSS PRODUCT

Find the unit vector $\mathbf{i}_n$ perpendicular in the right-hand sense to the vectors shown in Figure 1-10.

$$\mathbf{A} = -\mathbf{i}_x + \mathbf{i}_y + \mathbf{i}_z, \quad \mathbf{B} = \mathbf{i}_x - \mathbf{i}_y + \mathbf{i}_z$$

What is the angle between $\mathbf{A}$ and $\mathbf{B}$?

SOLUTION

The cross product $\mathbf{A} \times \mathbf{B}$ is perpendicular to both $\mathbf{A}$ and $\mathbf{B}$

$$\mathbf{A} \times \mathbf{B} = \det \begin{vmatrix} \mathbf{i}_x & \mathbf{i}_y & \mathbf{i}_z \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix} = 2(\mathbf{i}_x + \mathbf{i}_y)$$

The unit vector $\mathbf{i}_n$ is in this direction but it must have a magnitude of unity

$$\mathbf{i}_n = \frac{\mathbf{A} \times \mathbf{B}}{|\mathbf{A} \times \mathbf{B}|} = \frac{1}{\sqrt{2}} (\mathbf{i}_x + \mathbf{i}_y)$$

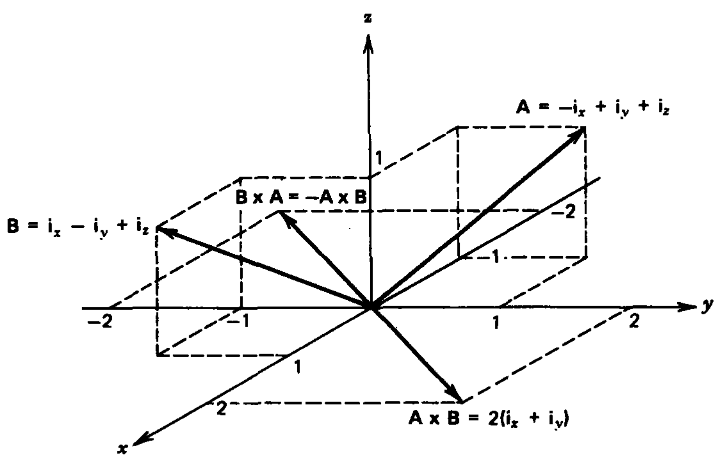

Figure 1-10 The cross product between the two vectors in Example 1-3.

The angle between **A** and **B** is found using (12) as

$$\sin\theta=\frac{|\mathbf{A}\times\mathbf{B}|}{AB}=\frac{2\sqrt{2}}{\sqrt{3}\sqrt{3}}$$
$$=\tfrac{2}{3}\sqrt{2}\Rightarrow\theta=70.5^\circ \text{ or } 109.5^\circ$$

The ambiguity in solutions can be resolved by using the dot product of (11)

$$\cos\theta=\frac{\mathbf{A}\cdot\mathbf{B}}{AB}=\frac{-1}{\sqrt{3}\sqrt{3}}=-\tfrac{1}{3}\Rightarrow\theta=109.5^\circ$$

1-3 THE GRADIENT AND THE DEL OPERATOR

1-3-1 The Gradient

Often we are concerned with the properties of a scalar field $f(x, y, z)$ around a particular point. The chain rule of differentiation then gives us the incremental change df in f for a small change in position from (x, y, z) to $(x+dx, y+dy, z+dz)$:

$$df=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy+\frac{\partial f}{\partial z}dz \tag{1}$$

If the general differential distance vector $\mathbf{dl}$ is defined as

$$\mathbf{dl}=dx\,\mathbf{i}_x+dy\,\mathbf{i}_y+dz\,\mathbf{i}_z \tag{2}$$

(1) can be written as the dot product:

$$df=\left(\frac{\partial f}{\partial x}\mathbf{i}_x+\frac{\partial f}{\partial y}\mathbf{i}_y+\frac{\partial f}{\partial z}\mathbf{i}_z\right)\cdot\mathbf{dl}$$
$$=\operatorname{grad} f\cdot\mathbf{dl} \tag{3}$$

where the spatial derivative terms in brackets are defined as the gradient of f:

$$\operatorname{grad} f=\nabla f=\frac{\partial f}{\partial x}\mathbf{i}_x+\frac{\partial f}{\partial y}\mathbf{i}_y+\frac{\partial f}{\partial z}\mathbf{i}_z \tag{4}$$

The symbol ∇ with the gradient term is introduced as a general vector operator, termed the del operator:

$$\nabla=\mathbf{i}_x\frac{\partial}{\partial x}+\mathbf{i}_y\frac{\partial}{\partial y}+\mathbf{i}_z\frac{\partial}{\partial z} \tag{5}$$

By itself the del operator is meaningless, but when it premultiplies a scalar function, the gradient operation is defined. We will soon see that the dot and cross products between the del operator and a vector also define useful operations.

With these definitions, the change in f of (3) can be written as

$$df = \nabla f \cdot \mathbf{dl} = |\nabla f|\, dl \cos\theta \tag{6}$$

where θ is the angle between ∇f and the position vector $\mathbf{dl}$. The direction that maximizes the change in the function f is when $\mathbf{dl}$ is colinear with $\nabla f(\theta = 0)$. The gradient thus has the direction of maximum change in f. Motions in the direction along lines of constant f have $\theta = \pi/2$ and thus by definition $df = 0$.

1-3-2 Curvilinear Coordinates

(a) Cylindrical

The gradient of a scalar function is defined for any coordinate system as that vector function that when dotted with $\mathbf{dl}$ gives df. In cylindrical coordinates the differential change in $f(\mathrm{r}, \phi, z)$ is

$$df = \frac{\partial f}{\partial \mathrm{r}}\, d\mathrm{r} + \frac{\partial f}{\partial \phi}\, d\phi + \frac{\partial f}{\partial z}\, dz \tag{7}$$

The differential distance vector is

$$\mathbf{dl} = d\mathrm{r}\mathbf{i}_\mathrm{r} + \mathrm{r}\, d\phi\, \mathbf{i}_\phi + dz\, \mathbf{i}_z \tag{8}$$

so that the gradient in cylindrical coordinates is

$$df = \nabla f \cdot \mathbf{dl} \Rightarrow \nabla f = \frac{\partial f}{\partial \mathrm{r}}\mathbf{i}_\mathrm{r} + \frac{1}{\mathrm{r}}\frac{\partial f}{\partial \phi}\mathbf{i}_\phi + \frac{\partial f}{\partial z}\mathbf{i}_z \tag{9}$$

(b) Spherical

Similarly in spherical coordinates the distance vector is

$$\mathbf{dl} = dr\, \mathbf{i}_r + r\, d\theta\, \mathbf{i}_\theta + r \sin\theta\, d\phi\, \mathbf{i}_\phi \tag{10}$$

with the differential change of $f(r, \theta, \phi)$ as

$$df = \frac{\partial f}{\partial r}\, dr + \frac{\partial f}{\partial \theta}\, d\theta + \frac{\partial f}{\partial \phi}\, d\phi = \nabla f \cdot \mathbf{dl} \tag{11}$$

Using (10) in (11) gives the gradient in spherical coordinates as

$$\nabla f = \frac{\partial f}{\partial r}\mathbf{i}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{i}_\theta + \frac{1}{r \sin\theta}\frac{\partial f}{\partial \phi}\mathbf{i}_\phi \tag{12}$$

EXAMPLE 1-4 GRADIENT

Find the gradient of each of the following functions where a and b are constants:

(a) $f = ax^2y + by^3z$

SOLUTION

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i}_x + \frac{\partial f}{\partial y}\mathbf{i}_y + \frac{\partial f}{\partial z}\mathbf{i}_z$$
$$= 2axy\mathbf{i}_x + (ax^2 + 3by^2z)\mathbf{i}_y + by^3\mathbf{i}_z$$

(b) $f = ar^2 \sin\phi + brz \cos 2\phi$

SOLUTION

$$\nabla f = \frac{\partial f}{\partial r}\mathbf{i}_r + \frac{1}{r}\frac{\partial f}{\partial \phi}\mathbf{i}_\phi + \frac{\partial f}{\partial z}\mathbf{i}_z$$
$$= (2ar \sin\phi + bz \cos 2\phi)\mathbf{i}_r$$
$$+ (ar \cos\phi - 2bz \sin 2\phi)\mathbf{i}_\phi + br \cos 2\phi \mathbf{i}_z$$

(c) $f = \frac{a}{r} + br \sin\theta \cos\phi$

SOLUTION

$$\nabla f = \frac{\partial f}{\partial r}\mathbf{i}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{i}_\theta + \frac{1}{r \sin\theta}\frac{\partial f}{\partial \phi}\mathbf{i}_\phi$$
$$= \left(-\frac{a}{r^2} + b \sin\theta \cos\phi\right)\mathbf{i}_r + b \cos\theta \cos\phi \mathbf{i}_\theta - b \sin\phi \mathbf{i}_\phi$$

1-3-3 The Line Integral

In Section 1-2-4 we motivated the use of the dot product through the definition of incremental work as depending only on the component of force $\mathbf{F}$ in the direction of an object's differential displacement $\mathbf{dl}$. If the object moves along a path, the total work is obtained by adding up the incremental works along each small displacement on the path as in Figure 1-11. If we break the path into N small displacements

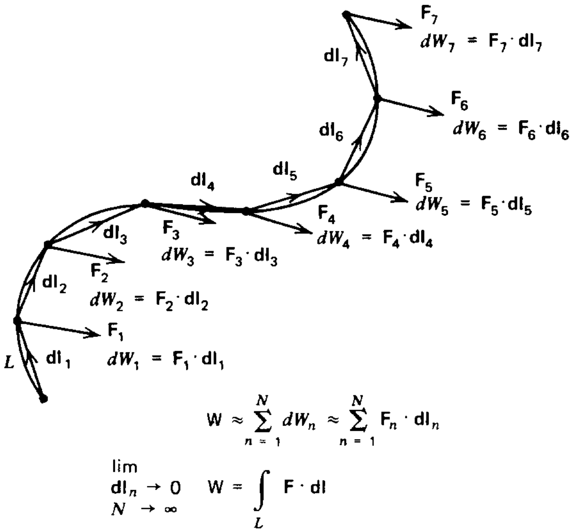

Figure 1-11 The total work in moving a body over a path is approximately equal to the sum of incremental works in moving the body each small incremental distance **dl**. As the differential distances approach zero length, the summation becomes a line integral and the result is exact.

$\mathbf{dl}_1, \mathbf{dl}_2, \ldots, \mathbf{dl}_N$, the work performed is approximately

$$W \approx \mathbf{F}_1 \cdot \mathbf{dl}_1 + \mathbf{F}_2 \cdot \mathbf{dl}_2 + \mathbf{F}_3 \cdot \mathbf{dl}_3 + \cdots + \mathbf{F}_N \cdot \mathbf{dl}_N$$
$$\approx \sum_{n=1}^{N} \mathbf{F}_n \cdot \mathbf{dl}_n \tag{13}$$

The result becomes exact in the limit as N becomes large with each displacement $\mathbf{dl}_n$ becoming infinitesimally small:

$$W = \lim_{\substack{N \to \infty \\ \mathbf{dl}_n \to 0}} \sum_{n=1}^{N} \mathbf{F}_n \cdot \mathbf{dl}_n = \int_L \mathbf{F} \cdot \mathbf{dl} \tag{14}$$

In particular, let us integrate (3) over a path between the two points a and b in Figure 1-12a:

$$\int_a^b df = f|_b - f|_a = \int_a^b \nabla f \cdot \mathbf{dl} \tag{15}$$

Because df is an exact differential, its line integral depends only on the end points and not on the shape of the contour itself. Thus, all of the paths between a and b in Figure 1-12a have the same line integral of ∇f, no matter what the function f may be. If the contour is a closed path so that $a = b$, as in

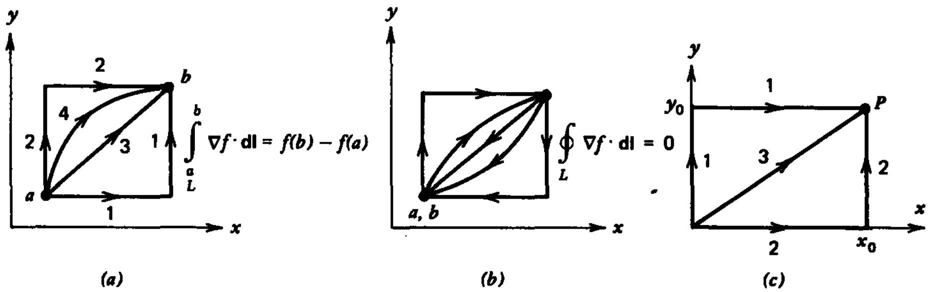

Figure 1-12 The component of the gradient of a function integrated along a line contour depends only on the end points and not on the contour itself. (*a*) Each of the contours have the same starting and ending points at *a* and *b* so that they all have the same line integral of ∇f. (*b*) When all the contours are closed with the same beginning and ending point at *a*, the line integral of ∇f is zero. (*c*) The line integral of the gradient of the function in Example (1-5) from the origin to the point *P* is the same for all paths.

Figure 1-12*b*, then (15) is zero:

$$\oint_L \nabla f \cdot \mathbf{dl} = f|_a - f|_a = 0 \tag{16}$$

where we indicate that the path is closed by the small circle in the integral sign $\oint$. The line integral of the gradient of a function around a closed path is zero.

EXAMPLE 1-5 LINE INTEGRAL

For $f = x^2 y$, verify (15) for the paths shown in Figure 1-12*c* between the origin and the point *P* at (x_0, y_0).

SOLUTION

The total change in f between 0 and *P* is

$$\int_0^P df = f|_P - f|_0 = x_0^2 y_0$$

From the line integral along path 1 we find

$$\int_0^P \nabla f \cdot \mathbf{dl} = \int_{\substack{y=0\\x=0}}^{y_0} \underbrace{\frac{\partial f}{\partial y}}_{x^2}\nearrow^{0} dy + \int_{\substack{x=0\\y=y_0}}^{x_0} \underbrace{\frac{\partial f}{\partial x}}_{2xy} dx = x_0^2 y_0$$

Similarly, along path 2 we also obtain

$$\int_0^P \nabla f \cdot d\mathbf{l} = \int_{\substack{x=0\\y=0}}^{x_0} \underbrace{\frac{\partial f}{\partial x}}_{2xy}^{\,0} dx + \int_{\substack{y=0\\x=x_0}}^{y_0} \underbrace{\frac{\partial f}{\partial y}}_{x^2} dy = x_0^2 y_0$$

while along path 3 we must relate x and y along the straight line as

$$y = \frac{y_0}{x_0} x \Rightarrow dy = \frac{y_0}{x_0} dx$$

to yield

$$\int_0^P \nabla f \cdot d\mathbf{l} = \int_0^P \left(\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \right) = \int_{x=0}^{x_0} \frac{3y_0 x^2}{x_0} dx = x_0^2 y_0$$

1-4 FLUX AND DIVERGENCE

If we measure the total mass of fluid entering the volume in Figure 1-13 and find it to be less than the mass leaving, we know that there must be an additional source of fluid within the pipe. If the mass leaving is less than that entering, then

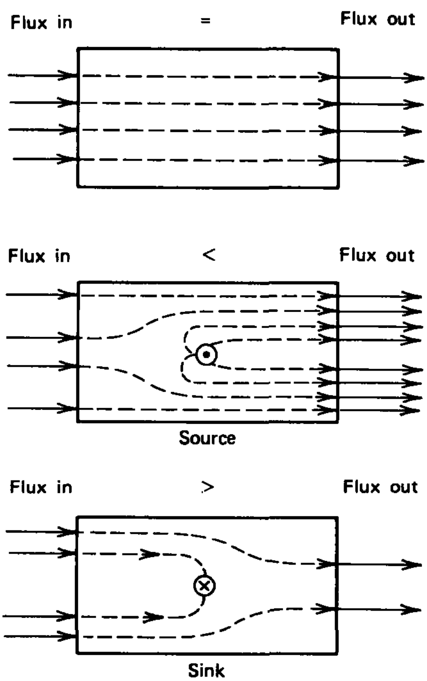

Figure 1-13 The net flux through a closed surface tells us whether there is a source or sink within an enclosed volume.

there is a sink (or drain) within the volume. In the absence of sources or sinks, the mass of fluid leaving equals that entering so the flow lines are continuous. Flow lines originate at a source and terminate at a sink.

1-4-1 Flux

We are illustrating with a fluid analogy what is called the flux Φ of a vector **A** through a closed surface:

$$\Phi = \oint_S \mathbf{A} \cdot \mathbf{dS} \tag{1}$$

The differential surface element **dS** is a vector that has magnitude equal to an incremental area on the surface but points in the direction of the outgoing unit normal **n** to the surface S, as in Figure 1-14. Only the component of **A** perpendicular to the surface contributes to the flux, as the tangential component only results in flow of the vector **A** along the surface and not through it. A positive contribution to the flux occurs if **A** has a component in the direction of **dS** out from the surface. If the normal component of **A** points into the volume, we have a negative contribution to the flux.

If there is no source for **A** within the volume V enclosed by the surface S, all the flux entering the volume equals that leaving and the net flux is zero. A source of **A** within the volume generates more flux leaving than entering so that the flux is positive ($\Phi > 0$) while a sink has more flux entering than leaving so that $\Phi < 0$.

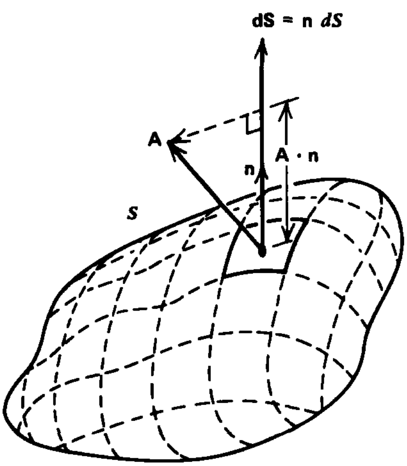

Figure 1-14 The flux of a vector **A** through the closed surface S is given by the surface integral of the component of **A** perpendicular to the surface S. The differential vector surface area element **dS** is in the direction of the unit normal **n**.

Thus we see that the sign and magnitude of the net flux relates the quantity of a field through a surface to the sources or sinks of the vector field within the enclosed volume.

1-4-2 Divergence

We can be more explicit about the relationship between the rate of change of a vector field and its sources by applying (1) to a volume of differential size, which for simplicity we take to be rectangular in Figure 1-15. There are three pairs of plane parallel surfaces perpendicular to the coordinate axes so that (1) gives the flux as

$$\begin{aligned}\Phi = &\int_1 A_x(x)\,dy\,dz - \int_{1'} A_x(x-\Delta x)\,dy\,dz\\ &+\int_2 A_y(y+\Delta y)\,dx\,dz - \int_{2'} A_y(y)\,dx\,dz\\ &+\int_3 A_z(z+\Delta z)\,dx\,dy - \int_{3'} A_z(z)\,dx\,dy \qquad (2)\end{aligned}$$

where the primed surfaces are differential distances behind the corresponding unprimed surfaces. The minus signs arise because the outgoing normals on the primed surfaces point in the negative coordinate directions.

Because the surfaces are of differential size, the components of **A** are approximately constant along each surface so that the surface integrals in (2) become pure

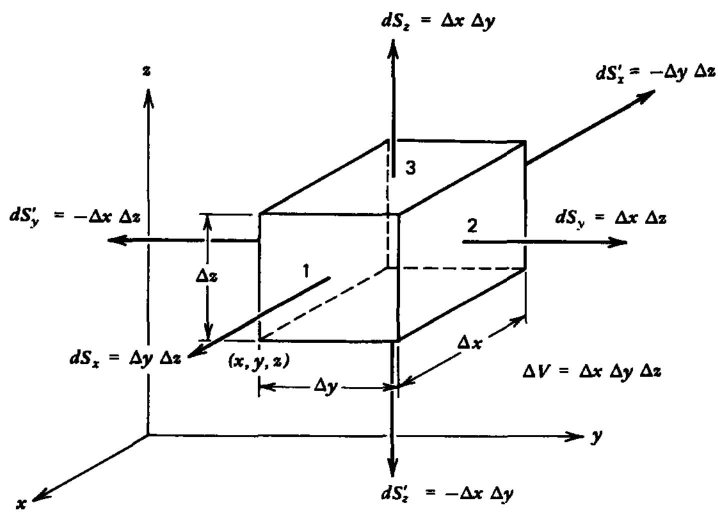

Figure 1-15 Infinitesimal rectangular volume used to define the divergence of a vector.

multiplications of the component of **A** perpendicular to the surface and the surface area. The flux then reduces to the form

$$\Phi \approx \left(\frac{[A_x(x) - A_x(x-\Delta x)]}{\Delta x} + \frac{[A_y(y+\Delta y) - A_y(y)]}{\Delta y} + \frac{[A_z(z+\Delta z) - A_z(z)]}{\Delta z}\right) \Delta x\, \Delta y\, \Delta z \tag{3}$$

We have written (3) in this form so that in the limit as the volume becomes infinitesimally small, each of the bracketed terms defines a partial derivative

$$\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0 \\ \Delta z \to 0}} \Phi = \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right) \Delta V \tag{4}$$

where $\Delta V = \Delta x\, \Delta y\, \Delta z$ is the volume enclosed by the surface S.

The coefficient of ΔV in (4) is a scalar and is called the divergence of **A**. It can be recognized as the dot product between the vector del operator of Section 1-3-1 and the vector **A**:

$$\text{div}\, \mathbf{A} = \nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \tag{5}$$

1-4-3 Curvilinear Coordinates

In cylindrical and spherical coordinates, the divergence operation is not simply the dot product between a vector and the del operator because the directions of the unit vectors are a function of the coordinates. Thus, derivatives of the unit vectors have nonzero contributions. It is easiest to use the generalized definition of the divergence independent of the coordinate system, obtained from (1)–(5) as

$$\nabla \cdot \mathbf{A} = \lim_{\Delta V \to 0} \frac{\oint_S \mathbf{A} \cdot d\mathbf{S}}{\Delta V} \tag{6}$$

(a) Cylindrical Coordinates

In cylindrical coordinates we use the small volume shown in Figure 1-16*a* to evaluate the net flux as

$$\begin{aligned}\Phi = \oint_S \mathbf{A} \cdot d\mathbf{S} &= \int_1 (\mathrm{r} + \Delta \mathrm{r}) A_{\mathrm{r}}\big|_{\mathrm{r}+\Delta\mathrm{r}}\, d\phi\, dz - \int_{1'} \mathrm{r} A_{\mathrm{r}}\big|_{\mathrm{r}}\, d\phi\, dz \\ &\quad + \int_2 A_\phi\big|_{\phi+\Delta\phi}\, d\mathrm{r}\, dz - \int_{2'} A_\phi\big|_\phi\, d\mathrm{r}\, dz \\ &\quad + \int_3 \mathrm{r} A_z\big|_{z+\Delta z}\, d\mathrm{r}\, d\phi - \int_{3'} \mathrm{r} A_z\big|_z\, d\mathrm{r}\, d\phi\end{aligned} \tag{7}$$

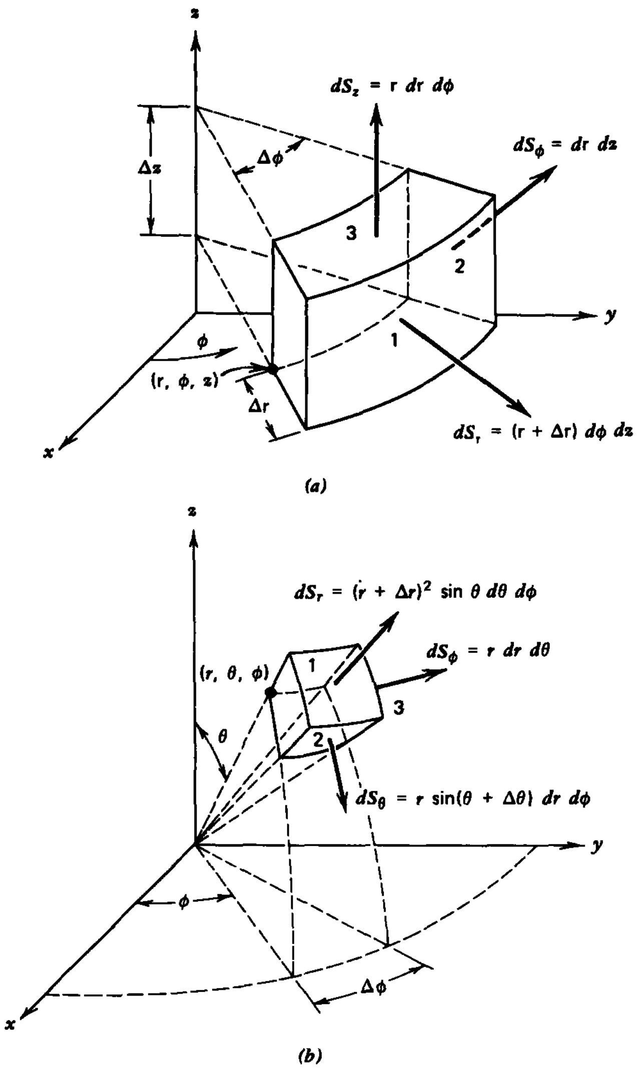

Figure 1-16 Infinitesimal volumes used to define the divergence of a vector in (*a*) cylindrical and (*b*) spherical geometries.

Again, because the volume is small, we can treat it as approximately rectangular with the components of **A** approximately constant along each face. Then factoring out the volume $\Delta V = r\Delta r\,\Delta\phi\,\Delta z$ in (7),

$$\Phi \approx \left(\frac{[(r+\Delta r)A_{r|_{r+\Delta r}} - rA_{r|_r}]}{r\,\Delta r} + \frac{[A_{\phi|_{\phi+\Delta\phi}} - A_{\phi|_\phi}]}{r\,\Delta\phi} + \frac{[A_{z|_{z+\Delta z}} - A_{z|_z}]}{\Delta z} \right) r\Delta r\,\Delta\phi\,\Delta z \tag{8}$$

lets each of the bracketed terms become a partial derivative as the differential lengths approach zero and (8) becomes an exact relation. The divergence is then

$$\nabla \cdot \mathbf{A} = \lim_{\substack{\Delta r \to 0 \\ \Delta \phi \to 0 \\ \Delta z \to 0}} \frac{\oint_S \mathbf{A} \cdot \mathbf{dS}}{\Delta V} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} \tag{9}$$

(b) Spherical Coordinates

Similar operations on the spherical volume element $\Delta V = r^2 \sin\theta\, \Delta r\, \Delta\theta\, \Delta\phi$ in Figure 1-16*b* defines the net flux through the surfaces:

$$\begin{aligned}\Phi &= \oint_S \mathbf{A} \cdot \mathbf{dS} \\ &\approx \left(\frac{[(r+\Delta r)^2 A_r|_{r+\Delta r} - r^2 A_r|_r]}{r^2\, \Delta r} \right. \\ &\quad + \frac{[A_\theta|_{\theta+\Delta\theta} \sin(\theta+\Delta\theta) - A_\theta|_\theta \sin\theta]}{r \sin\theta\, \Delta\theta} \\ &\quad \left. + \frac{[A_\phi|_{\phi+\Delta\phi} - A_\phi|_\phi]}{r \sin\theta\, \Delta\phi} \right) r^2 \sin\theta\, \Delta r\, \Delta\theta\, \Delta\phi \end{aligned} \tag{10}$$

The divergence in spherical coordinates is then

$$\begin{aligned}\nabla \cdot \mathbf{A} &= \lim_{\substack{\Delta r \to 0 \\ \Delta \theta \to 0 \\ \Delta \phi \to 0}} \frac{\oint_S \mathbf{A} \cdot \mathbf{dS}}{\Delta V} \\ &= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi} \end{aligned} \tag{11}$$

1-4-4 The Divergence Theorem

If we now take many adjoining incremental volumes of any shape, we form a macroscopic volume V with enclosing surface S as shown in Figure 1-17*a*. However, each interior common surface between incremental volumes has the flux leaving one volume (positive flux contribution) just entering the adjacent volume (negative flux contribution) as in Figure 1-17*b*. The net contribution to the flux for the surface integral of (1) is zero for all interior surfaces. Nonzero contributions to the flux are obtained only for those surfaces which bound the outer surface S of V. Although the surface contributions to the flux using (1) cancel for all interior volumes, the flux obtained from (4) in terms of the divergence operation for

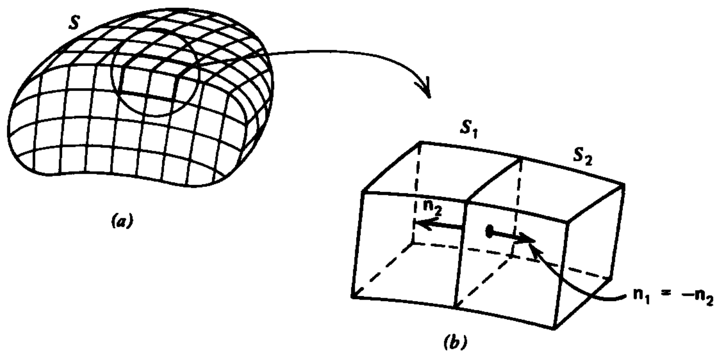

Figure 1-17 Nonzero contributions to the flux of a vector are only obtained across those surfaces that bound the outside of a volume. (a) Within the volume the flux leaving one incremental volume just enters the adjacent volume where (*b*) the outgoing normals to the common surface separating the volumes are in opposite directions.

each incremental volume add. By adding all contributions from each differential volume, we obtain the divergence theorem:

$$\Phi = \oint_S \mathbf{A} \cdot \mathbf{dS} = \lim_{\substack{N \to \infty \\ \Delta V_n \to 0}} \sum_{n=1}^{\infty} (\nabla \cdot \mathbf{A})\, \Delta V_n = \int_V \nabla \cdot \mathbf{A}\, dV \quad (12)$$

where the volume V may be of macroscopic size and is enclosed by the outer surface S. This powerful theorem converts a surface integral into an equivalent volume integral and will be used many times in our development of electromagnetic field theory.

EXAMPLE 1-6 THE DIVERGENCE THEOREM

Verify the divergence theorem for the vector

$$\mathbf{A} = x\mathbf{i}_x + y\mathbf{i}_y + z\mathbf{i}_z = r\mathbf{i}_r$$

by evaluating both sides of (12) for the rectangular volume shown in Figure 1-18.

SOLUTION

The volume integral is easier to evaluate as the divergence of $\mathbf{A}$ is a constant

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = 3$$

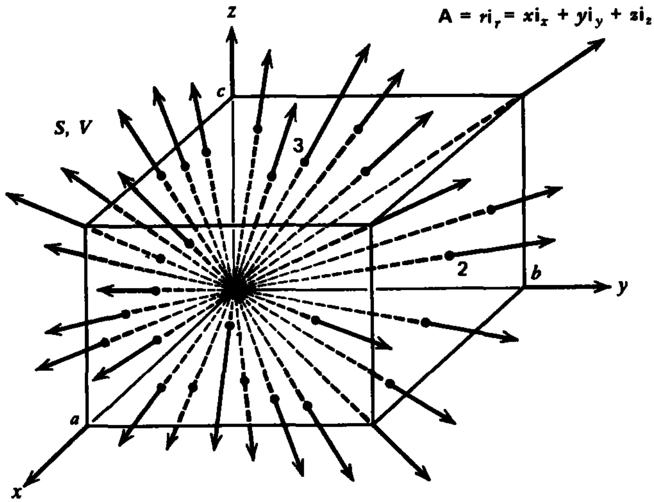

Figure 1-18 The divergence theorem is verified in Example 1-6 for the radial vector through a rectangular volume.

(In spherical coordinates $\nabla \cdot \mathbf{A} = (1/r^2)(\partial/\partial r)(r^3) = 3$) so that the volume integral in (12) is

$$\int_V \nabla \cdot \mathbf{A}\, dV = 3abc$$

The flux passes through the six plane surfaces shown:

$$\Phi = \oint_S \mathbf{A} \cdot \mathbf{dS} = \int_1 \underbrace{A_x(a)}_{a}\, dy\, dz - \int_{1'} \underbrace{A_x(0)}_{0}\, dy\, dz$$

$$+ \int_2 \underbrace{A_y(b)}_{b}\, dx\, dz - \int_{2'} \underbrace{A_y(0)}_{0}\, dx\, dz$$

$$+ \int_3 \underbrace{A_z(c)}_{c}\, dx\, dy - \int_{3'} \underbrace{A_z(0)}_{0}\, dx\, dy = 3abc$$

which verifies the divergence theorem.

1.5 THE CURL AND STOKES' THEOREM

1-5-1 Curl

We have used the example of work a few times previously to motivate particular vector and integral relations. Let us do so once again by considering the line integral of a vector around a closed path called the circulation:

$$C = \oint_L \mathbf{A} \cdot d\mathbf{l} \tag{1}$$

where if C is the work, $\mathbf{A}$ would be the force. We evaluate (1) for the infinitesimal rectangular contour in Figure 1-19a:

$$C = \int_{\underset{1}{x}}^{x+\Delta x} A_x(y)\,dx + \int_{\underset{2}{y}}^{y+\Delta y} A_y(x+\Delta x)\,dy + \int_{\underset{3}{x+\Delta x}}^{x} A_x(y+\Delta y)\,dx$$

$$+ \int_{\underset{4}{y+\Delta y}}^{y} A_y(x)\,dy \tag{2}$$

The components of $\mathbf{A}$ are approximately constant over each differential sized contour leg so that (2) is approximated as

$$C \approx \left(\frac{[A_x(y) - A_x(y+\Delta y)]}{\Delta y} + \frac{[A_y(x+\Delta x) - A_y(x)]}{\Delta x} \right) \Delta x\, \Delta y \tag{3}$$

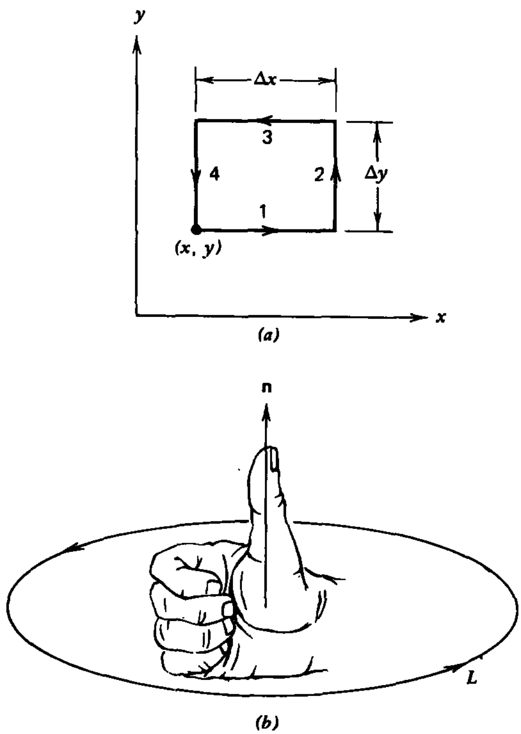

Figure 1-19 (a) Infinitesimal rectangular contour used to define the circulation. (b) The right-hand rule determines the positive direction perpendicular to a contour.

where terms are factored so that in the limit as Δx and Δy become infinitesimally small, (3) becomes exact and the bracketed terms define partial derivatives:

$$\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0 \\ \Delta S_z = \Delta x \Delta y}} C = \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \Delta S_z \tag{4}$$

The contour in Figure 1-19*a* could just have as easily been in the *xz* or *yz* planes where (4) would equivalently become

$$C = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) \Delta S_x \quad (yz \text{ plane})$$

$$C = \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) \Delta S_y \quad (xz \text{ plane}) \tag{5}$$

by simple positive permutations of x, y, and z.

The partial derivatives in (4) and (5) are just components of the cross product between the vector del operator of Section 1-3-1 and the vector **A**. This operation is called the curl of **A** and it is also a vector:

$$\begin{aligned} \text{curl}\,\mathbf{A} = \nabla \times \mathbf{A} &= \det \begin{vmatrix} \mathbf{i}_x & \mathbf{i}_y & \mathbf{i}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} \\ &= \mathbf{i}_x\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \mathbf{i}_y\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) \\ &\quad + \mathbf{i}_z\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \end{aligned} \tag{6}$$

The cyclical permutation of (x, y, z) allows easy recall of (6) as described in Section 1-2-5.

In terms of the curl operation, the circulation for any differential sized contour can be compactly written as

$$C = (\nabla \times \mathbf{A}) \cdot \mathbf{dS} \tag{7}$$

where $\mathbf{dS} = \mathbf{n}\, dS$ is the area element in the direction of the normal vector **n** perpendicular to the plane of the contour in the sense given by the right-hand rule in traversing the contour, illustrated in Figure 1-19*b*. Curling the fingers on the right hand in the direction of traversal around the contour puts the thumb in the direction of the normal **n**.

For a physical interpretation of the curl it is convenient to continue to use a fluid velocity field as a model although the general results and theorems are valid for any vector field. If

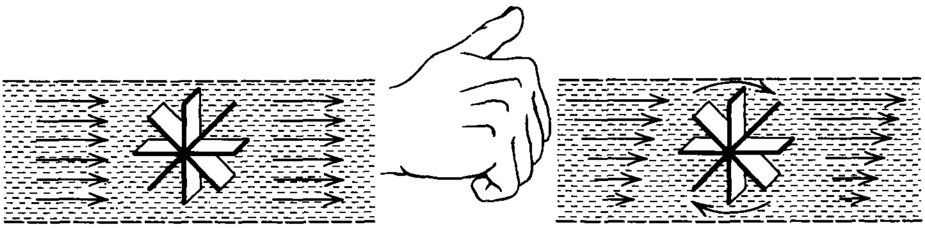

Figure 1-20 A fluid with a velocity field that has a curl tends to turn the paddle wheel. The curl component found is in the same direction as the thumb when the fingers of the right hand are curled in the direction of rotation.

a small paddle wheel is imagined to be placed without disturbance in a fluid flow, the velocity field is said to have circulation, that is, a nonzero curl, if the paddle wheel rotates as illustrated in Figure 1-20. The curl component found is in the direction of the axis of the paddle wheel.

1-5-2 The Curl for Curvilinear Coordinates

A coordinate independent definition of the curl is obtained using (7) in (1) as

$$(\nabla\times\mathbf{A})_n=\lim_{dS_n\to 0}\frac{\oint_L \mathbf{A}\cdot d\mathbf{l}}{dS_n} \tag{8}$$

where the subscript n indicates the component of the curl perpendicular to the contour. The derivation of the curl operation (8) in cylindrical and spherical coordinates is straightforward but lengthy.

(a) Cylindrical Coordinates

To express each of the components of the curl in cylindrical coordinates, we use the three orthogonal contours in Figure 1-21. We evaluate the line integral around contour a:

$$\begin{aligned}\oint_a \mathbf{A}\cdot d\mathbf{l}&=\int_z^{z-\Delta z}A_z(\phi)\,dz+\int_\phi^{\phi+\Delta\phi}A_\phi(z-\Delta z)\,\mathrm{r}\,d\phi\\&\quad+\int_{z-\Delta z}^{z}A_z(\phi+\Delta\phi)\,dz+\int_{\phi+\Delta\phi}^{\phi}A_\phi(z)\,\mathrm{r}\,d\phi\\&\approx\left(\frac{[A_z(\phi+\Delta\phi)-A_z(\phi)]}{\mathrm{r}\Delta\phi}-\frac{[A_\phi(z)-A_\phi(z-\Delta z)]}{\Delta z}\right)\mathrm{r}\,\Delta\phi\,\Delta z\end{aligned} \tag{9}$$

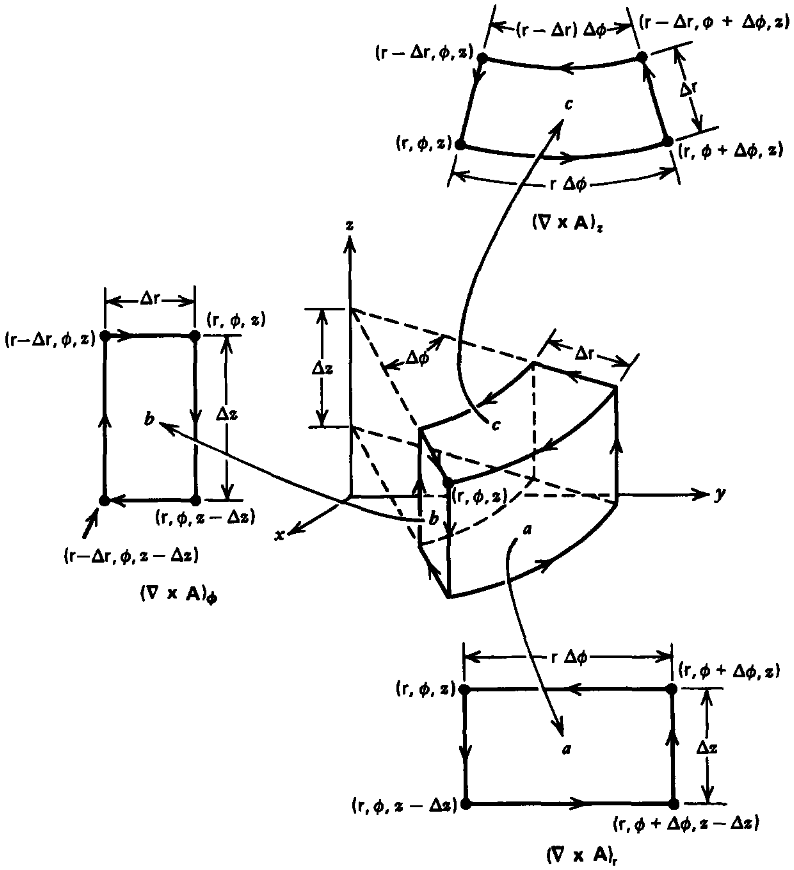

Figure 1-21 Incremental contours along cylindrical surface area elements used to calculate each component of the curl of a vector in cylindrical coordinates.

to find the radial component of the curl as

$$(\nabla\times\mathbf{A})_r=\lim_{\substack{\Delta\phi\to 0\\ \Delta z\to 0}}\frac{\oint_a \mathbf{A}\cdot d\mathbf{l}}{r\,\Delta\phi\,\Delta z}=\frac{1}{r}\frac{\partial A_z}{\partial\phi}-\frac{\partial A_\phi}{\partial z} \tag{10}$$

We evaluate the line integral around contour b:

$$\oint_b \mathbf{A}\cdot d\mathbf{l}=\int_{r-\Delta r}^{r} A_r(z)\,dr+\int_{z}^{z-\Delta z} A_z(r)\,dz+\int_{r}^{r-\Delta r} A_r(z-\Delta z)\,dr$$
$$+\int_{z-\Delta z}^{z} A_z(r-\Delta r)\,dz$$
$$\approx\left(\frac{[A_r(z)-A_r(z-\Delta z)]}{\Delta z}-\frac{[A_z(r)-A_z(r-\Delta r)]}{\Delta r}\right)\Delta r\,\Delta z \tag{11}$$

to find the ϕ component of the curl,

$$(\nabla\times\mathbf{A})_\phi=\lim_{\substack{\Delta r\to 0\\ \Delta z\to 0}}\frac{\oint_b \mathbf{A}\cdot d\mathbf{l}}{\Delta r\,\Delta z}=\left(\frac{\partial A_r}{\partial z}-\frac{\partial A_z}{\partial r}\right) \tag{12}$$

The z component of the curl is found using contour c:

$$\begin{aligned}\oint_c \mathbf{A}\cdot d\mathbf{l}&=\int_{r-\Delta r}^{r} A_{r|_\phi}\,dr+\int_{\phi}^{\phi+\Delta\phi} rA_{\phi|_r}\,d\phi+\int_{r}^{r-\Delta r} A_{r|_{\phi+\Delta\phi}}\,dr\\&\quad+\int_{\phi+\Delta\phi}^{\phi}(r-\Delta r)A_{\phi|_{r-\Delta r}}\,d\phi\\&\approx\left(\frac{[rA_{\phi|_r}-(r-\Delta r)A_{\phi|_{r-\Delta r}}]}{r\Delta r}-\frac{[A_{r|_{\phi+\Delta\phi}}-A_{r|_\phi}]}{r\Delta\phi}\right)r\,\Delta r\,\Delta\phi\end{aligned} \tag{13}$$

to yield

$$(\nabla\times\mathbf{A})_z=\lim_{\substack{\Delta r\to 0\\ \Delta\phi\to 0}}\frac{\oint_c \mathbf{A}\cdot d\mathbf{l}}{r\,\Delta r\,\Delta\phi}=\frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\phi)-\frac{\partial A_r}{\partial\phi}\right) \tag{14}$$

The curl of a vector in cylindrical coordinates is thus

$$\begin{aligned}\nabla\times\mathbf{A}&=\left(\frac{1}{r}\frac{\partial A_z}{\partial\phi}-\frac{\partial A_\phi}{\partial z}\right)\mathbf{i}_r+\left(\frac{\partial A_r}{\partial z}-\frac{\partial A_z}{\partial r}\right)\mathbf{i}_\phi\\&\quad+\frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\phi)-\frac{\partial A_r}{\partial\phi}\right)\mathbf{i}_z\end{aligned} \tag{15}$$

(b) Spherical Coordinates

Similar operations on the three incremental contours for the spherical element in Figure 1-22 give the curl in spherical coordinates. We use contour a for the radial component of the curl:

$$\begin{aligned}\oint_a \mathbf{A}\cdot d\mathbf{l}&=\int_{\phi}^{\phi+\Delta\phi} A_{\phi|_\theta} r\sin\theta\,d\phi+\int_{\theta}^{\theta-\Delta\theta} rA_{\theta|_{\phi+\Delta\phi}}\,d\theta\\&\quad+\int_{\phi+\Delta\phi}^{\phi} r\sin(\theta-\Delta\theta)A_{\phi|_{\theta-\Delta\theta}}\,d\phi+\int_{\theta-\Delta\theta}^{\theta} rA_{\theta|_\phi}\,d\theta\\&\approx\left(\frac{[A_{\phi|_\theta}\sin\theta-A_{\phi|_{\theta-\Delta\theta}}\sin(\theta-\Delta\theta)]}{r\sin\theta\,\Delta\theta}\right.\\&\quad\left.-\frac{[A_{\theta|_{\phi+\Delta\phi}}-A_{\theta|_\phi}]}{r\sin\theta\,\Delta\phi}\right)r^2\sin\theta\,\Delta\theta\,\Delta\phi\end{aligned} \tag{16}$$

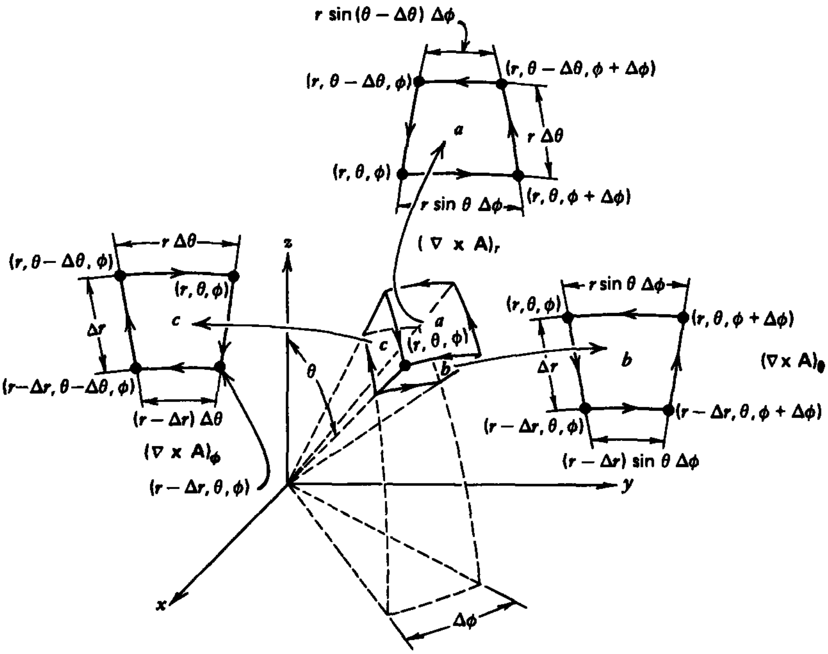

Figure 1-22 Incremental contours along spherical surface area elements used to calculate each component of the curl of a vector in spherical coordinates.

to obtain

$$(\nabla\times\mathbf{A})_r=\lim_{\substack{\Delta\theta\to 0\\ \Delta\phi\to 0}}\frac{\oint_a \mathbf{A}\cdot d\mathbf{l}}{r^2\sin\theta\,\Delta\theta\,\Delta\phi}=\frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(A_\phi\sin\theta)-\frac{\partial A_\theta}{\partial\phi}\right)\tag{17}$$

The θ component is found using contour b:

$$\begin{aligned}\oint_b \mathbf{A}\cdot d\mathbf{l}=&\int_r^{r-\Delta r} A_r|_\phi\,dr+\int_\phi^{\phi+\Delta\phi}(r-\Delta r)A_\phi|_{r-\Delta r}\sin\theta\,d\phi\\&+\int_{r-\Delta r}^{r} A_r|_{\phi+\Delta\phi}\,dr+\int_{\phi+\Delta\phi}^{\phi} rA_\phi|_r\sin\theta\,d\phi\\ \approx&\left(\frac{[A_r|_{\phi+\Delta\phi}-A_r|_\phi]}{r\sin\theta\,\Delta\phi}\right.\\&\left.-\frac{[rA_\phi|_r-(r-\Delta r)A_\phi|_{r-\Delta r}]}{r\,\Delta r}\right)r\sin\theta\,\Delta r\,\Delta\phi\end{aligned}\tag{18}$$

as

$$(\nabla\times\mathbf{A})_\theta=\lim_{\substack{\Delta r\to 0\\ \Delta\phi\to 0}}\frac{\oint_b \mathbf{A}\cdot d\mathbf{l}}{r\sin\theta\,\Delta r\,\Delta\phi}=\frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi}-\frac{\partial}{\partial r}(rA_\phi)\right) \tag{19}$$

The ϕ component of the curl is found using contour c:

$$\begin{aligned}\oint_c \mathbf{A}\cdot d\mathbf{l}&=\int_{\theta-\Delta\theta}^{\theta} rA_{\theta|_r}\,d\theta+\int_r^{r-\Delta r}A_{r|_\theta}\,dr\\&\quad+\int_\theta^{\theta-\Delta\theta}(r-\Delta r)A_{\theta|_{r-\Delta r}}\,d\theta+\int_{r-\Delta r}^{r}A_{r|_{\theta-\Delta\theta}}\,dr\\&\approx\left(\frac{[rA_{\theta|_r}-(r-\Delta r)A_{\theta|_{r-\Delta r}}]}{r\,\Delta r}-\frac{[A_{r|_\theta}-A_{r|_{\theta-\Delta\theta}}]}{r\,\Delta\theta}\right)r\,\Delta r\,\Delta\theta\end{aligned} \tag{20}$$

as

$$(\nabla\times\mathbf{A})_\phi=\lim_{\substack{\Delta r\to 0\\ \Delta\theta\to 0}}\frac{\oint_c \mathbf{A}\cdot d\mathbf{l}}{r\,\Delta r\,\Delta\theta}=\frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\theta)-\frac{\partial A_r}{\partial\theta}\right) \tag{21}$$

The curl of a vector in spherical coordinates is thus given from (17), (19), and (21) as

$$\begin{aligned}\nabla\times\mathbf{A}&=\frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(A_\phi\sin\theta)-\frac{\partial A_\theta}{\partial\phi}\right)\mathbf{i}_r\\&\quad+\frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi}-\frac{\partial}{\partial r}(rA_\phi)\right)\mathbf{i}_\theta\\&\quad+\frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\theta)-\frac{\partial A_r}{\partial\theta}\right)\mathbf{i}_\phi\end{aligned} \tag{22}$$

1-5-3 Stokes' Theorem

We now piece together many incremental line contours of the type used in Figures 1-19–1-21 to form a macroscopic surface S like those shown in Figure 1-23. Then each small contour generates a contribution to the circulation

$$dC=(\nabla\times\mathbf{A})\cdot d\mathbf{S} \tag{23}$$

so that the total circulation is obtained by the sum of all the small surface elements

$$C=\int_S(\nabla\times\mathbf{A})\cdot d\mathbf{S} \tag{24}$$

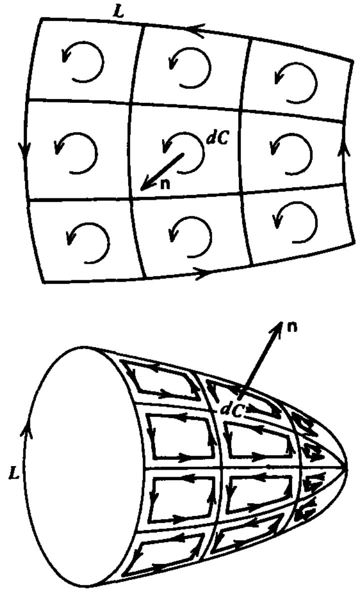

Figure 1-23 Many incremental line contours distributed over any surface, have nonzero contribution to the circulation only along those parts of the surface on the boundary contour L.

Each of the terms of (23) are equivalent to the line integral around each small contour. However, all interior contours share common sides with adjacent contours but which are twice traversed in opposite directions yielding no net line integral contribution, as illustrated in Figure 1-23. Only those contours with a side on the open boundary L have a nonzero contribution. The total result of adding the contributions for all the contours is Stokes' theorem, which converts the line integral over the bounding contour L of the outer edge to a surface integral over any area S bounded by the contour

$$\oint_L \mathbf{A} \cdot \mathbf{dl} = \int_S (\nabla \times \mathbf{A}) \cdot \mathbf{dS} \tag{25}$$

Note that there are an infinite number of surfaces that are bounded by the same contour L. Stokes' theorem of (25) is satisfied for all these surfaces.

EXAMPLE 1-7 STOKES' THEOREM

Verify Stokes' theorem of (25) for the circular bounding contour in the xy plane shown in Figure 1-24 with a vector

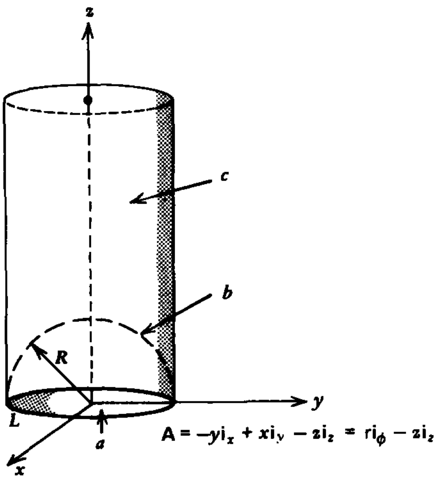

Figure 1-24 Stokes' theorem for the vector given in Example 1-7 can be applied to any surface that is bounded by the same contour L.

field

$$\mathbf{A} = -y\mathbf{i}_x + x\mathbf{i}_y - z\mathbf{i}_z = r\mathbf{i}_\phi - z\mathbf{i}_z$$

Check the result for the (a) flat circular surface in the xy plane, (b) for the hemispherical surface bounded by the contour, and (c) for the cylindrical surface bounded by the contour.

SOLUTION

For the contour shown

$$\mathbf{dl} = R\, d\phi\, \mathbf{i}_\phi$$

so that

$$\mathbf{A} \cdot \mathbf{dl} = R^2\, d\phi$$

where on L, $r = R$. Then the circulation is

$$C = \oint_L \mathbf{A} \cdot \mathbf{dl} = \int_0^{2\pi} R^2\, d\phi = 2\pi R^2$$

The z component of $\mathbf{A}$ had no contribution because $\mathbf{dl}$ was entirely in the xy plane.

The curl of $\mathbf{A}$ is

$$\nabla \times \mathbf{A} = \mathbf{i}_z\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) = 2\mathbf{i}_z$$

(a) For the circular area in the plane of the contour, we have that

$$\int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S} = 2\int_S dS_z = 2\pi R^2$$

which agrees with the line integral result.

(b) For the hemispherical surface

$$\int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S} = \int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi} 2\mathbf{i}_z \cdot \mathbf{i}_r R^2 \sin\theta \, d\theta \, d\phi$$

From Table 1-2 we use the dot product relation

$$\mathbf{i}_z \cdot \mathbf{i}_r = \cos\theta$$

which again gives the circulation as

$$C = \int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi} R^2 \sin 2\theta \, d\theta \, d\phi = -2\pi R^2 \frac{\cos 2\theta}{2}\bigg|_{\theta=0}^{\pi/2} = 2\pi R^2$$

(c) Similarly, for the cylindrical surface, we only obtain nonzero contributions to the surface integral at the upper circular area that is perpendicular to $\nabla \times \mathbf{A}$. The integral is then the same as part (a) as $\nabla \times \mathbf{A}$ is independent of z.

1-5-4 Some Useful Vector Identities

The curl, divergence, and gradient operations have some simple but useful properties that are used throughout the text.

(a) The Curl of the Gradient is Zero [$\nabla \times (\nabla f) = 0$]

We integrate the normal component of the vector $\nabla \times (\nabla f)$ over a surface and use Stokes' theorem

$$\int_S \nabla \times (\nabla f) \cdot d\mathbf{S} = \oint_L \nabla f \cdot d\mathbf{l} = 0 \tag{26}$$

where the zero result is obtained from Section 1-3-3, that the line integral of the gradient of a function around a closed path is zero. Since the equality is true for any surface, the vector coefficient of $d\mathbf{S}$ in (26) must be zero

$$\nabla \times (\nabla f) = 0$$

The identity is also easily proved by direct computation using the determinantal relation in Section 1-5-1 defining the curl operation:

$$\nabla\times(\nabla f)=\det\begin{vmatrix}\mathbf{i}_x & \mathbf{i}_y & \mathbf{i}_z\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} & \dfrac{\partial f}{\partial z}\end{vmatrix}$$

$$=\mathbf{i}_x\left(\frac{\partial^2 f}{\partial y\partial z}-\frac{\partial^2 f}{\partial z\partial y}\right)+\mathbf{i}_y\left(\frac{\partial^2 f}{\partial z\partial x}-\frac{\partial^2 f}{\partial x\partial z}\right)+\mathbf{i}_z\left(\frac{\partial^2 f}{\partial x\partial y}-\frac{\partial^2 f}{\partial y\partial x}\right)=0 \tag{28}$$

Each bracketed term in (28) is zero because the order of differentiation does not matter.

(b) The Divergence of the Curl of a Vector is Zero $[\nabla\cdot(\nabla\times\mathbf{A})=0]$

One might be tempted to apply the divergence theorem to the surface integral in Stokes' theorem of (25). However, the divergence theorem requires a closed surface while Stokes' theorem is true in general for an open surface. Stokes' theorem for a closed surface requires the contour L to shrink to zero giving a zero result for the line integral. The divergence theorem applied to the closed surface with vector $\nabla\times\mathbf{A}$ is then

$$\oint_S \nabla\times\mathbf{A}\cdot d\mathbf{S}=0\Rightarrow\int_V \nabla\cdot(\nabla\times\mathbf{A})\,dV=0\Rightarrow\nabla\cdot(\nabla\times\mathbf{A})=0 \tag{29}$$

which proves the identity because the volume is arbitrary.

More directly we can perform the required differentiations

$$\nabla\cdot(\nabla\times\mathbf{A})$$
$$=\frac{\partial}{\partial x}\left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right)+\frac{\partial}{\partial y}\left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right)+\frac{\partial}{\partial z}\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)$$
$$=\left(\frac{\partial^2 A_z}{\partial x\partial y}-\frac{\partial^2 A_z}{\partial y\partial x}\right)+\left(\frac{\partial^2 A_x}{\partial y\partial z}-\frac{\partial^2 A_x}{\partial z\partial y}\right)+\left(\frac{\partial^2 A_y}{\partial z\partial x}-\frac{\partial^2 A_y}{\partial x\partial z}\right)=0 \tag{30}$$

where again the order of differentiation does not matter.

PROBLEMS

Section 1-1

1. Find the area of a circle in the xy plane centered at the origin using:

(a) rectangular coordinates $x^2+y^2=a^2$ **(Hint:**

$$\int\sqrt{a^2-x^2}\,dx=\tfrac{1}{2}[x\sqrt{a^2-x^2}+a^2\sin^{-1}(x/a)])$$