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Electromagnetic Field Theory: A Problem Solving Approach

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chapter 9

radiation

In low-frequency electric circuits and along transmission lines, power is guided from a source to a load along highly conducting wires with the fields predominantly confined to the region around the wires. At very high frequencies these wires become antennas as this power can radiate away into space without the need of any guiding structure.

9-1 THE RETARDED POTENTIALS

9-1-1 Nonhomogeneous Wave Equations

Maxwell's equations in complete generality are

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (1)$$

$$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t} \quad (2)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (3)$$

$$\nabla \cdot \mathbf{D} = \rho_f \quad (4)$$

In our development we will use the following vector identities

$$\nabla \times (\nabla V) = 0 \quad (5)$$

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0 \quad (6)$$

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \quad (7)$$

where \mathbf{A} and V can be any functions but in particular will be the magnetic vector potential and electric scalar potential, respectively.

Because in (3) the magnetic field has no divergence, the identity in (6) allows us to again define the vector potential \mathbf{A} as we had for quasi-statics in Section 5-4:

$$\mathbf{B} = \nabla \times \mathbf{A} \quad (8)$$

so that Faraday's law in (1) can be rewritten as

$$\nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = 0 \quad (9)$$

Then (5) tells us that any curl-free vector can be written as the gradient of a scalar so that (9) becomes

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla V \quad (10)$$

where we introduce the negative sign on the right-hand side so that V becomes the electric potential in a static situation when \mathbf{A} is independent of time. We solve (10) for the electric field and with (8) rewrite (2) for linear dielectric media ($\mathbf{D} = \epsilon \mathbf{E}$, $\mathbf{B} = \mu \mathbf{H}$):

$$\nabla \times (\nabla \times \mathbf{A}) = \mu \mathbf{J}_f + \frac{1}{c^2} \left[-\nabla \left(\frac{\partial V}{\partial t} \right) - \frac{\partial^2 \mathbf{A}}{\partial t^2} \right], \quad c^2 = \frac{1}{\epsilon \mu} \quad (11)$$

The vector identity of (7) allows us to reduce (11) to

$$\nabla^2 \mathbf{A} - \nabla \left[\nabla \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial V}{\partial t} \right] - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu \mathbf{J}_f \quad (12)$$

Thus far, we have only specified the curl of \mathbf{A} in (8). The Helmholtz theorem discussed in Section 5-4-1 told us that to uniquely specify the vector potential we must also specify the divergence of \mathbf{A} . This is called setting the gauge. Examining (12) we see that if we set

$$\nabla \cdot \mathbf{A} = -\frac{1}{c^2} \frac{\partial V}{\partial t} \quad (13)$$

the middle term on the left-hand side of (12) becomes zero so that the resulting relation between \mathbf{A} and \mathbf{J}_f is the non-homogeneous vector wave equation:

$$\nabla^2 \mathbf{A} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu \mathbf{J}_f \quad (14)$$

The condition of (13) is called the Lorentz gauge. Note that for static conditions, $\nabla \cdot \mathbf{A} = 0$, which is the value also picked in Section 5-4-2 for the magneto-quasi-static field. With (14) we can solve for \mathbf{A} when the current distribution \mathbf{J}_f is given and then use (13) to solve for V . The scalar potential can also be found directly by using (10) in Gauss's law of (4) as

$$\nabla^2 V + \frac{\partial}{\partial t} (\nabla \cdot \mathbf{A}) = \frac{-\rho_f}{\epsilon} \quad (15)$$

The second term can be put in terms of V by using the Lorentz gauge condition of (13) to yield the scalar wave equation:

$$\nabla^2 V - \frac{1}{c^2} \frac{\partial^2 V}{\partial t^2} = \frac{-\rho_f}{\epsilon} \quad (16)$$

Note again that for static situations this relation reduces to Poisson's equation, the governing equation for the quasi-static electric potential.

9-1-2 Solutions to the Wave Equation

We see that the three scalar equations of (14) (one equation for each vector component) and that of (16) are in the same form. If we can thus find the general solution to any one of these equations, we know the general solution to all of them.

As we had earlier proceeded for quasi-static fields, we will find the solution to (16) for a point charge source. Then the solution for any charge distribution is obtained using superposition by integrating the solution for a point charge over all incremental charge elements.

In particular, consider a stationary point charge at $r=0$ that is an arbitrary function of time $Q(t)$. By symmetry, the resulting potential can only be a function of r so that (16) becomes

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial V}{\partial r} \right) - \frac{1}{c^2} \frac{\partial^2 V}{\partial t^2} = 0, \quad r > 0 \quad (17)$$

where the right-hand side is zero because the charge density is zero everywhere except at $r=0$. By multiplying (17) through by r and realizing that

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r^2 \frac{\partial V}{\partial r} \right) = \frac{\partial^2}{\partial r^2} (rV) \quad (18)$$

we rewrite (17) as a homogeneous wave equation in the variable (rV) :

$$\frac{\partial^2}{\partial r^2} (rV) - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} (rV) = 0 \quad (19)$$

which we know from Section 7-3-2 has solutions

$$rV = f_+ \left(t - \frac{r}{c} \right) + f_- \left(t + \frac{r}{c} \right) \quad (20)$$

We throw out the negatively traveling wave solution as there are no sources for $r > 0$ so that all waves emanate radially outward from the point charge at $r=0$. The arbitrary function f_+ is evaluated by realizing that as $r \rightarrow 0$ there can be no propagation delay effects so that the potential should approach the quasi-static Coulomb potential of a point charge:

$$\lim_{r \rightarrow 0} V = \frac{Q(t)}{4\pi\epsilon r} \Rightarrow f_+(t) = \frac{Q(t)}{4\pi\epsilon} \quad (21)$$

The potential due to a point charge is then obtained from (20) and (21) replacing time t with the retarded time $t - r/c$:

$$V(r, t) = \frac{Q(t - r/c)}{4\pi\epsilon r} \quad (22)$$

The potential at time t depends not on the present value of charge but on the charge value a propagation time r/c earlier when the wave now received was launched.

The potential due to an arbitrary volume distribution of charge $\rho_f(t)$ is obtained by replacing $Q(t)$ with the differential charge element $\rho_f(t) dV$ and integrating over the volume of charge:

$$V(r, t) = \int_{\text{all charge}} \frac{\rho_f(t - r_{QP}/c)}{4\pi\epsilon r_{QP}} dV \quad (23)$$

where r_{QP} is the distance between the charge as a source at point Q and the field point at P .

The vector potential in (14) is in the same direction as the current density \mathbf{J}_f . The solution for \mathbf{A} can be directly obtained from (23) realizing that each component of \mathbf{A} obeys the same equation as (16) if we replace ρ_f/ϵ by $\mu\mathbf{J}_f$:

$$\mathbf{A}(r, t) = \int_{\text{all current}} \frac{\mu\mathbf{J}_f(t - r_{QP}/c)}{4\pi r_{QP}} dV \quad (24)$$

9-2 RADIATION FROM POINT DIPOLES

9-2-1 The Electric Dipole

The simplest building block for a transmitting antenna is that of a uniform current flowing along a conductor of incremental length dl as shown in Figure 9-1. We assume that this current varies sinusoidally with time as

$$i(t) = \text{Re} (\hat{I} e^{j\omega t}) \quad (1)$$

Because the current is discontinuous at the ends, charge must be deposited there being of opposite sign at each end [$q(t) = \text{Re} (\hat{Q} e^{j\omega t})$]:

$$i(t) = \pm \frac{dq}{dt} \Rightarrow \hat{I} = \pm j\omega \hat{Q}, \quad z = \pm \frac{dl}{2} \quad (2)$$

This forms an electric dipole with moment

$$\mathbf{p} = q dl \mathbf{i}_z \quad (3)$$

If we can find the potentials and fields from this simple element, the solution for any current distribution is easily found by superposition.

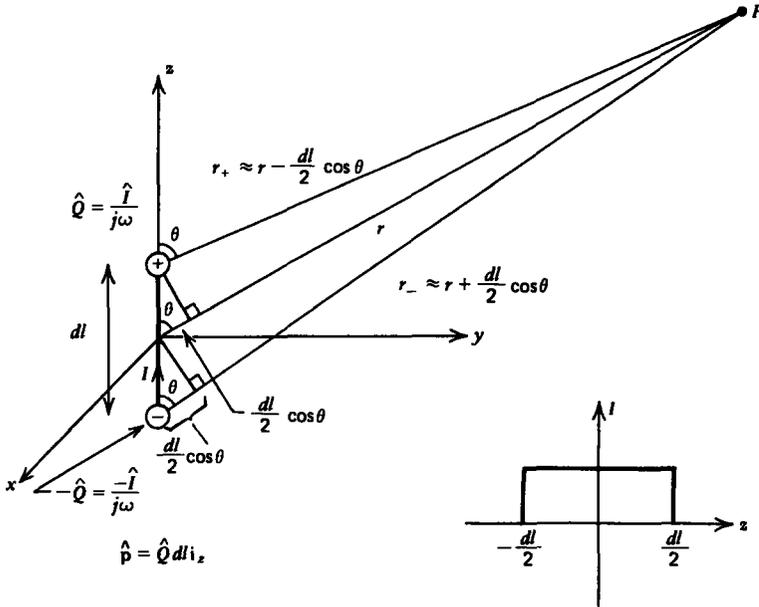


Figure 9-1 A point dipole antenna is composed of a very short uniformly distributed current-carrying wire. Because the current is discontinuous at the ends, equal magnitude but opposite polarity charges accumulate there forming an electric dipole.

By symmetry, the vector potential cannot depend on the angle ϕ ,

$$A_z = \text{Re} [\hat{A}_z(r, \theta) e^{j\omega t}] \tag{4}$$

and must be in the same direction as the current:

$$A_z(r, t) = \text{Re} \left[\int_{-dl/2}^{+dl/2} \frac{\mu \hat{I} e^{j[\omega(t - r_{QP}/c)]}}{4\pi r_{QP}} dz \right] \tag{5}$$

Because the dipole is of infinitesimal length, the distance from the dipole to any field point is just the spherical radial distance r and is constant for all points on the short wire. Then the integral in (5) reduces to a pure multiplication to yield

$$\hat{A}_z = \frac{\mu \hat{I} dl}{4\pi r} e^{-jkr}, \quad A_z(r, t) = \text{Re} [\hat{A}_z(r) e^{j\omega t}] \tag{6}$$

where we again introduce the wavenumber $k = \omega/c$ and neglect writing the sinusoidal time dependence present in all field and source quantities. The spherical components of \hat{A}_z

are ($\mathbf{i}_z = \mathbf{i}_r \cos \theta - \mathbf{i}_\theta \sin \theta$):

$$\hat{A}_r = \hat{A}_z \cos \theta, \quad \hat{A}_\theta = -\hat{A}_z \sin \theta, \quad \hat{A}_\phi = 0 \quad (7)$$

Once the vector potential is known, the electric and magnetic fields are most easily found from

$$\begin{aligned} \hat{\mathbf{H}} &= \frac{1}{\mu} \nabla \times \hat{\mathbf{A}}, & \mathbf{H}(\mathbf{r}, t) &= \text{Re} [\hat{\mathbf{H}}(\mathbf{r}, \theta) e^{j\omega t}] \\ \hat{\mathbf{E}} &= \frac{1}{j\omega\epsilon} \nabla \times \hat{\mathbf{H}}, & \mathbf{E}(\mathbf{r}, t) &= \text{Re} [\hat{\mathbf{E}}(\mathbf{r}, \theta) e^{j\omega t}] \end{aligned} \quad (8)$$

Before we find these fields, let's examine an alternate approach.

9-2-2 Alternate Derivation Using the Scalar Potential

It was easiest to find the vector potential for the point electric dipole because the integration in (5) reduced to a simple multiplication. The scalar potential is due solely to the opposite point charges at each end of the dipole,

$$\hat{V} = \frac{\hat{Q}}{4\pi\epsilon} \left(\frac{e^{-jk r_+}}{r_+} - \frac{e^{-jk r_-}}{r_-} \right) \quad (9)$$

where r_+ and r_- are the distances from the respective dipole charges to any field point, as shown in Figure 9-1. Just as we found for the quasi-static electric dipole in Section 3-1-1, we cannot let r_+ and r_- equal r as a zero potential would result. As we showed in Section 3-1-1, a first-order correction must be made, where

$$\begin{aligned} r_+ &\approx r - \frac{dl}{2} \cos \theta \\ r_- &\approx r + \frac{dl}{2} \cos \theta \end{aligned} \quad (10)$$

so that (9) becomes

$$\hat{V} \approx \frac{\hat{Q}}{4\pi\epsilon r} e^{-jkr} \left(\frac{e^{jk(dl/2)\cos\theta}}{\left(1 - \frac{dl}{2r}\cos\theta\right)} - \frac{e^{-jk(dl/2)\cos\theta}}{\left(1 + \frac{dl}{2r}\cos\theta\right)} \right) \quad (11)$$

Because the dipole length dl is assumed much smaller than the field distance r and the wavelength, the phase factors in the exponentials are small so they and the $1/r$ dependence in the denominators can be expanded in a first-order Taylor

series to result in:

$$\begin{aligned} \lim_{\substack{k dl \ll 1 \\ dl/r \ll 1}} \hat{V} &\approx \frac{\hat{Q}}{4\pi\epsilon r} e^{-jkr} \left[\left(1 + j\frac{k dl}{2} \cos \theta\right) \left(1 + \frac{dl}{2r} \cos \theta\right) \right. \\ &\quad \left. - \left(1 - jk\frac{dl}{2} \cos \theta\right) \left(1 - \frac{dl}{2r} \cos \theta\right) \right] \\ &= \frac{\hat{Q} dl}{4\pi\epsilon r^2} e^{-jkr} \cos \theta (1 + jkr) \end{aligned} \quad (12)$$

When the frequency becomes very low so that the wavenumber also becomes small, (12) reduces to the quasi-static electric dipole potential found in Section 3-1-1 with dipole moment $\hat{p} = \hat{Q} dl$. However, we see that the radiation correction terms in (12) dominate at higher frequencies (large k) far from the dipole ($kr \gg 1$) so that the potential only dies off as $1/r$ rather than the quasi-static $1/r^2$. Using the relationships $\hat{Q} = \hat{I}/j\omega$ and $c = 1/\sqrt{\epsilon\mu}$, (12) could have been obtained immediately from (6) and (7) with the Lorentz gauge condition of Eq. (13) in Section 9-1-1:

$$\begin{aligned} \hat{V} &= \frac{-c^2}{j\omega} \nabla \cdot \hat{A} = \frac{-c^2}{j\omega} \left(\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \hat{A}_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\hat{A}_\theta \sin \theta) \right) \\ &= \frac{\mu \hat{I} dl c^2}{4\pi j\omega} \frac{(1 + jkr)}{r^2} e^{-jkr} \cos \theta \\ &= \frac{\hat{Q} dl}{4\pi\epsilon r^2} (1 + jkr) e^{-jkr} \cos \theta \end{aligned} \quad (13)$$

9-2-3 The Electric and Magnetic Fields

Using (6), the fields are directly found from (8) as

$$\begin{aligned} \hat{H} &= \frac{1}{\mu} \nabla \times \hat{A} \\ &= \hat{i}_\phi \frac{1}{\mu r} \left(\frac{\partial}{\partial r} (r \hat{A}_\theta) - \frac{\partial \hat{A}_r}{\partial \theta} \right) \\ &= -\hat{i}_\phi \frac{\hat{I} dl}{4\pi} k^2 \sin \theta \left(\frac{1}{jkr} + \frac{1}{(jkr)^2} \right) e^{-jkr} \end{aligned} \quad (14)$$

$$\begin{aligned}
\hat{\mathbf{E}} &= \frac{1}{j\omega\epsilon} \nabla \times \hat{\mathbf{H}} \\
&= \frac{1}{j\omega\epsilon} \left(\frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\hat{H}_\phi \sin \theta) \mathbf{i}_r - \frac{1}{r} \frac{\partial}{\partial r} (r \hat{H}_\phi) \mathbf{i}_\theta \right) \\
&= -\frac{\hat{I} dl k^2}{4\pi} \sqrt{\frac{\mu}{\epsilon}} \left\{ \mathbf{i}_r \left[2 \cos \theta \left(\frac{1}{(jkr)^2} + \frac{1}{(jkr)^3} \right) \right] \right. \\
&\quad \left. + \mathbf{i}_\theta \left[\sin \theta \left(\frac{1}{jkr} + \frac{1}{(jkr)^2} + \frac{1}{(jkr)^3} \right) \right] \right\} e^{-jkr} \quad (15)
\end{aligned}$$

Note that even this simple source generates a fairly complicated electromagnetic field. The magnetic field in (14) points purely in the ϕ direction as expected by the right-hand rule for a z -directed current. The term that varies as $1/r^2$ is called the induction field or near field for it predominates at distances close to the dipole and exists even at zero frequency. The new term, which varies as $1/r$, is called the radiation field since it dominates at distances far from the dipole and will be shown to be responsible for time-average power flow away from the source. The near field term does not contribute to power flow but is due to the stored energy in the magnetic field and thus results in reactive power.

The $1/r^3$ terms in (15) are just the electric dipole field terms present even at zero frequency and so are often called the electrostatic solution. They predominate at distances close to the dipole and thus are the near fields. The electric field also has an intermediate field that varies as $1/r^2$, but more important is the radiation field term in the \mathbf{i}_θ component, which varies as $1/r$. At large distances ($kr \gg 1$) this term dominates.

In the far field limit ($kr \gg 1$), the electric and magnetic fields are related to each other in the same way as for plane waves:

$$\lim_{kr \gg 1} \hat{E}_\theta = \sqrt{\frac{\mu}{\epsilon}} \hat{H}_\phi = \frac{\hat{E}_0}{jkr} \sin \theta e^{-jkr}, \quad \hat{E}_0 = -\frac{\hat{I} dl k^2}{4\pi} \sqrt{\frac{\mu}{\epsilon}} \quad (16)$$

The electric and magnetic fields are perpendicular and their ratio is equal to the wave impedance $\eta = \sqrt{\mu/\epsilon}$. This is because in the far field limit the spherical wavefronts approximate a plane.

9-2-4 Electric Field Lines

Outside the dipole the volume charge density is zero, which allows us to define an electric vector potential \mathbf{C} :

$$\nabla \cdot \mathbf{E} = 0 \Rightarrow \mathbf{E} = \nabla \times \mathbf{C} \quad (17)$$

Because the electric field in (15) only has r and θ components, \mathbf{C} must only have a ϕ component, $C_\phi(r, \theta)$:

$$\mathbf{E} = \nabla \times \mathbf{C} = \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta C_\phi) \mathbf{i}_r - \frac{1}{r} \frac{\partial}{\partial r} (r C_\phi) \mathbf{i}_\theta \quad (18)$$

We follow the same procedure developed in Section 4-4-3*b*, where the electric field lines are given by

$$\frac{dr}{r d\theta} = \frac{E_r}{E_\theta} = - \frac{\frac{\partial}{\partial \theta} (\sin \theta C_\phi)}{\sin \theta \frac{\partial}{\partial r} (r C_\phi)} \quad (19)$$

which can be rewritten as an exact differential,

$$\frac{\partial}{\partial r} (r \sin \theta C_\phi) dr + \frac{\partial}{\partial \theta} (r \sin \theta C_\phi) d\theta = 0 \Rightarrow d(r \sin \theta C_\phi) = 0 \quad (20)$$

so that the field lines are just lines of constant stream-function $r \sin \theta C_\phi$. C_ϕ is found by equating each vector component in (18) to the solution in (15):

$$\begin{aligned} \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta \hat{C}_\phi) &= \hat{E}_r = -\frac{\hat{I} dl k^2}{4\pi} \sqrt{\frac{\mu}{\epsilon}} \left[2 \cos \theta \left(\frac{1}{(jkr)^2} + \frac{1}{(jkr)^3} \right) \right] e^{-jkr} \\ -\frac{1}{r} \frac{\partial}{\partial r} (r \hat{C}_\phi) &= \hat{E}_\theta = -\frac{\hat{I} dl k^2}{4\pi} \sqrt{\frac{\mu}{\epsilon}} \left[\sin \theta \left(\frac{1}{(jkr)} + \frac{1}{(jkr)^2} + \frac{1}{(jkr)^3} \right) \right] e^{-jkr} \end{aligned} \quad (21)$$

which integrates to

$$\hat{C}_\phi = \frac{\hat{I} dl}{4\pi} \sqrt{\frac{\mu}{\epsilon}} \frac{\sin \theta}{r} \left(1 - \frac{j}{kr} \right) e^{-jkr} \quad (22)$$

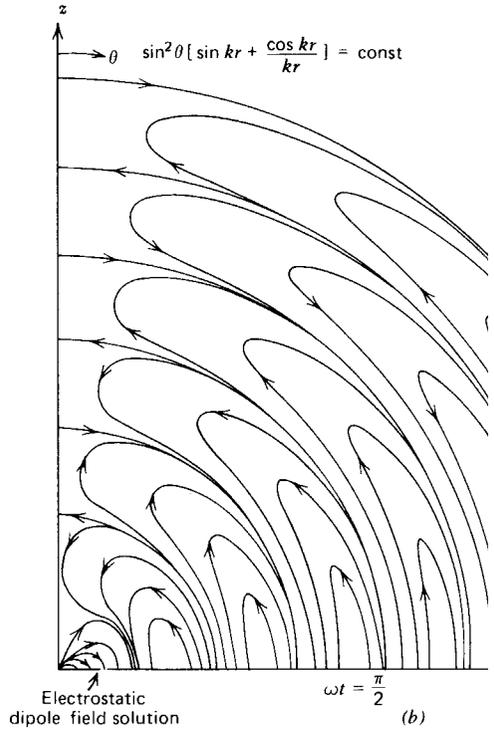
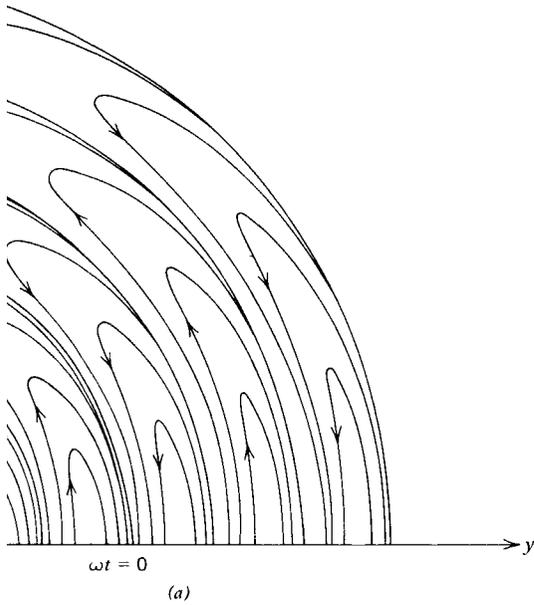
Then assuming \hat{I} is real, the instantaneous value of C_ϕ is

$$\begin{aligned} C_\phi &= \text{Re} (\hat{C}_\phi e^{j\omega t}) \\ &= \frac{\hat{I} dl}{4\pi} \sqrt{\frac{\mu}{\epsilon}} \frac{\sin \theta}{r} \left(\cos(\omega t - kr) + \frac{\sin(\omega t - kr)}{kr} \right) \end{aligned} \quad (23)$$

so that, omitting the constant amplitude factor in (23), the field lines are

$$r C_\phi \sin \theta = \text{const} \Rightarrow \sin^2 \theta \left(\cos(\omega t - kr) + \frac{\sin(\omega t - kr)}{kr} \right) = \text{const} \quad (24)$$

$$\frac{\sin kr}{kr} = \text{const}$$



The electric field lines for a point electric dipole at $\omega t = 0$ and $\omega t = \pi/2$.

These field lines are plotted in Figure 9-2 at two values of time. We can check our result with the static field lines for a dipole given in Section 3-1-1. Remembering that $k = \omega/c$, at low frequencies,

$$\lim_{\omega \rightarrow 0} \begin{cases} \cos(\omega t - kr) \approx 1 \\ \frac{\sin(\omega t - kr)}{kr} \approx \frac{(t - r/c)}{r/c} \approx \frac{t}{r/c} - 1 \end{cases} \quad (25)$$

so that, in the low-frequency limit at a fixed time, (24) approaches the result of Eq. (6) of Section 3-1-1:

$$\lim_{\omega \rightarrow 0} \sin^2 \theta \left(\frac{ct}{r} \right) = \text{const} \quad (26)$$

Note that the field lines near the dipole are those of a static dipole field, as drawn in Figure 3-2. In the far field limit

$$\lim_{kr \gg 1} \sin^2 \theta \cos(\omega t - kr) = \text{const} \quad (27)$$

the field lines repeat with period $\lambda = 2\pi/k$.

9-2-5 Radiation Resistance

Using the electric and magnetic fields of Section 9-2-3, the time-average power density is

$$\begin{aligned} \langle \mathbf{S} \rangle &= \frac{1}{2} \text{Re} (\hat{\mathbf{E}} \times \hat{\mathbf{H}}^*) \\ &= \text{Re} \left\{ \frac{|\hat{I} dl|^2 \eta k^4}{2(4\pi)^2} \left[-\mathbf{i}_\theta \sin 2\theta \left(-\frac{1}{(jkr)^3} + \frac{1}{(jkr)^5} \right) \right] \right. \\ &\quad \left. + \mathbf{i}_r \sin^2 \theta \left(-\frac{1}{(jkr)^2} + \frac{1}{(jkr)^5} \right) \right\} \\ &= \frac{1}{2} |\hat{I} dl|^2 \left(\frac{k}{4\pi} \right)^2 \frac{\eta}{r^2} \sin^2 \theta \mathbf{i}_r \\ &= \frac{1}{2} \frac{|\hat{E}_0|^2}{\eta} \frac{\sin^2 \theta}{(kr)^2} \mathbf{i}_r \end{aligned} \quad (28)$$

where \hat{E}_0 is defined in (16).

Only the far fields contributed to the time-average power flow. The near and intermediate fields contributed only imaginary terms in (28) representing reactive power.

The power density varies with the angle θ , being zero along the electric dipole's axis ($\theta = 0, \pi$) and maximum at right angles to it ($\theta = \pi/2$), illustrated by the radiation power pattern in Fig. 9-3. The strength of the power density is proportional to the length of the vector from the origin to the

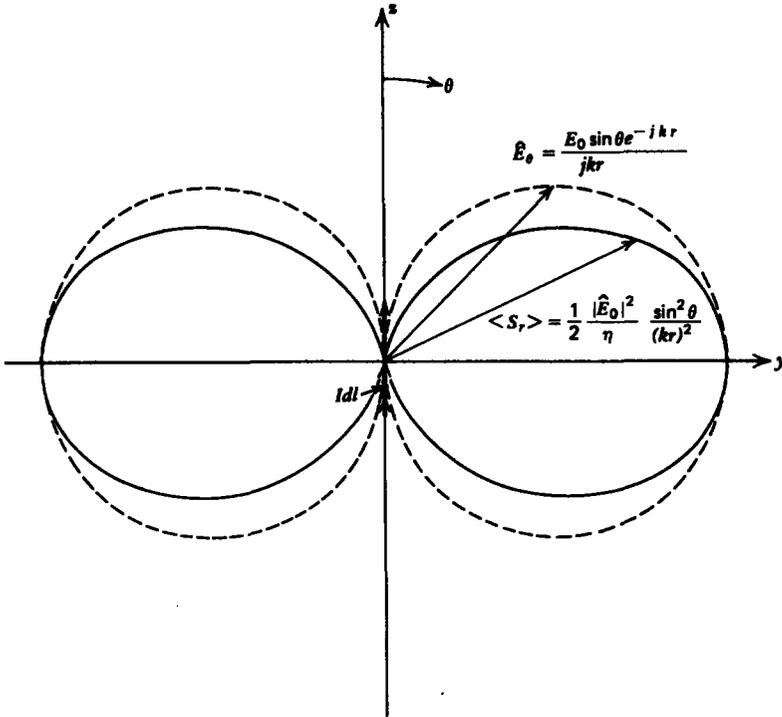


Figure 9-3 The strength of the electric field and power density due to a z -directed point dipole as a function of angle θ is proportional to the length of the vector from the origin to the radiation pattern.

radiation pattern. These directional properties are useful in beam steering, where the directions of power flow can be controlled.

The total time-average power radiated by the electric dipole is found by integrating the Poynting vector over a spherical surface at any radius r :

$$\begin{aligned}
 \langle P \rangle &= \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \langle S_r \rangle r^2 \sin \theta \, d\theta \, d\phi \\
 &= \frac{1}{2} |\hat{I} dl|^2 \left(\frac{k}{4\pi} \right)^2 \eta 2\pi \int_{\theta=0}^{\pi} \sin^3 \theta \, d\theta \\
 &= \frac{|\hat{I} dl|^2}{16\pi} \eta k^2 \left[-\frac{1}{3} \cos \theta (\sin^2 \theta + 2) \right]_0^{\pi} \\
 &= \frac{|\hat{I} dl|^2}{12\pi} \eta k^2
 \end{aligned} \tag{29}$$

As far as the dipole is concerned, this radiated power is lost in the same way as if it were dissipated in a resistance R ,

$$\langle P \rangle = \frac{1}{2} |\hat{I}|^2 R \quad (30)$$

where this equivalent resistance is called the radiation resistance:

$$R = \frac{\eta}{6\pi} (k dl)^2 = \frac{2\pi\eta}{3} \left(\frac{dl}{\lambda}\right)^2, \quad k = \frac{2\pi}{\lambda} \quad (31)$$

In free space $\eta_0 = \sqrt{\mu_0/\epsilon_0} \approx 120\pi$, the radiation resistance is

$$R_0 = 80\pi^2 \left(\frac{dl}{\lambda}\right)^2 \quad (\text{free space}) \quad (32)$$

These results are only true for point dipoles, where dl is much less than a wavelength ($dl/\lambda \ll 1$). This verifies the validity of the quasi-static approximation for geometries much smaller than a radiated wavelength, as the radiated power is then negligible.

If the current on a dipole is not constant but rather varies with z over the length, the only term that varies with z for the vector potential in (5) is $\hat{I}(z)$:

$$\hat{A}_z(r) = \text{Re} \left[\int_{-dl/2}^{+dl/2} \frac{\mu \hat{I}(z) e^{-jkr_{QP}}}{4\pi r_{QP}} dz \right] \approx \text{Re} \left[\frac{\mu e^{-jkr_{QP}}}{4\pi r_{QP}} \int_{-dl/2}^{+dl/2} \hat{I}(z) dz \right] \quad (33)$$

where, because the dipole is of infinitesimal length, the distance r_{QP} from any point on the dipole to any field point far from the dipole is essentially r , independent of z . Then, all further results for the electric and magnetic fields are the same as in Section 9-2-3 if we replace the actual dipole length dl by its effective length,

$$dl_{\text{eff}} = \frac{1}{I_0} \int_{-dl/2}^{+dl/2} \hat{I}(z) dz \quad (34)$$

where \hat{I}_0 is the terminal current feeding the center of the dipole.

Generally the current is zero at the open circuited ends, as for the linear distribution shown in Figure 9-4,

$$\hat{I}(z) = \begin{cases} I_0(1 - 2z/dl), & 0 \leq z \leq dl/2 \\ I_0(1 + 2z/dl), & -dl/2 \leq z \leq 0 \end{cases} \quad (35)$$

so that the effective length is half the actual length:

$$dl_{\text{eff}} = \frac{1}{I_0} \int_{-dl/2}^{+dl/2} \hat{I}(z) dz = \frac{dl}{2} \quad (36)$$

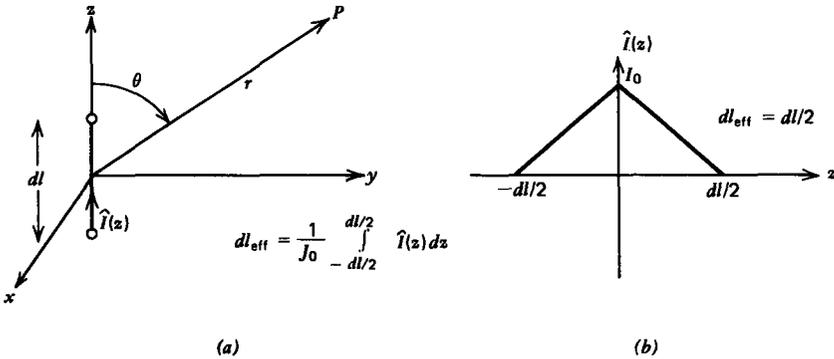


Figure 9-4 (a) If a point electric dipole has a nonuniform current distribution, the solutions are of the same form if we replace the actual dipole length dl by an effective length dl_{eff} . (b) For a triangular current distribution the effective length is half the true length.

Because the fields are reduced by half, the radiation resistance is then reduced by $\frac{1}{4}$:

$$R = \frac{2\pi\eta}{3} \left(\frac{dl_{\text{eff}}}{\lambda} \right)^2 = 20\pi^2 \sqrt{\frac{\mu_r}{\epsilon_r}} \left(\frac{dl}{\lambda} \right)^2 \tag{37}$$

In free space the relative permeability μ_r and relative permittivity ϵ_r are unity.

Note also that with a spatially dependent current distribution, a line charge distribution is found over the whole length of the dipole and not just on the ends:

$$\hat{\lambda} = -\frac{1}{j\omega} \frac{d\hat{I}}{dz} \tag{38}$$

For the linear current distribution described by (35), we see that:

$$\hat{\lambda} = \pm \frac{2I_0}{j\omega dl} \begin{cases} 0 \leq z \leq dl/2 \\ -dl/2 \leq z \leq 0 \end{cases} \tag{39}$$

9-2-6 Rayleigh Scattering (or why is the sky blue?)

If a plane wave electric field $\text{Re} [E_0 e^{j\omega t} \mathbf{i}_x]$ is incident upon an atom that is much smaller than the wavelength, the induced dipole moment also contributes to the resultant field, as illustrated in Figure 9-5. The scattered power is perpendicular to the induced dipole moment. Using the dipole model developed in Section 3-1-4, where a negative spherical electron cloud of radius R_0 with total charge $-Q$ surrounds a fixed

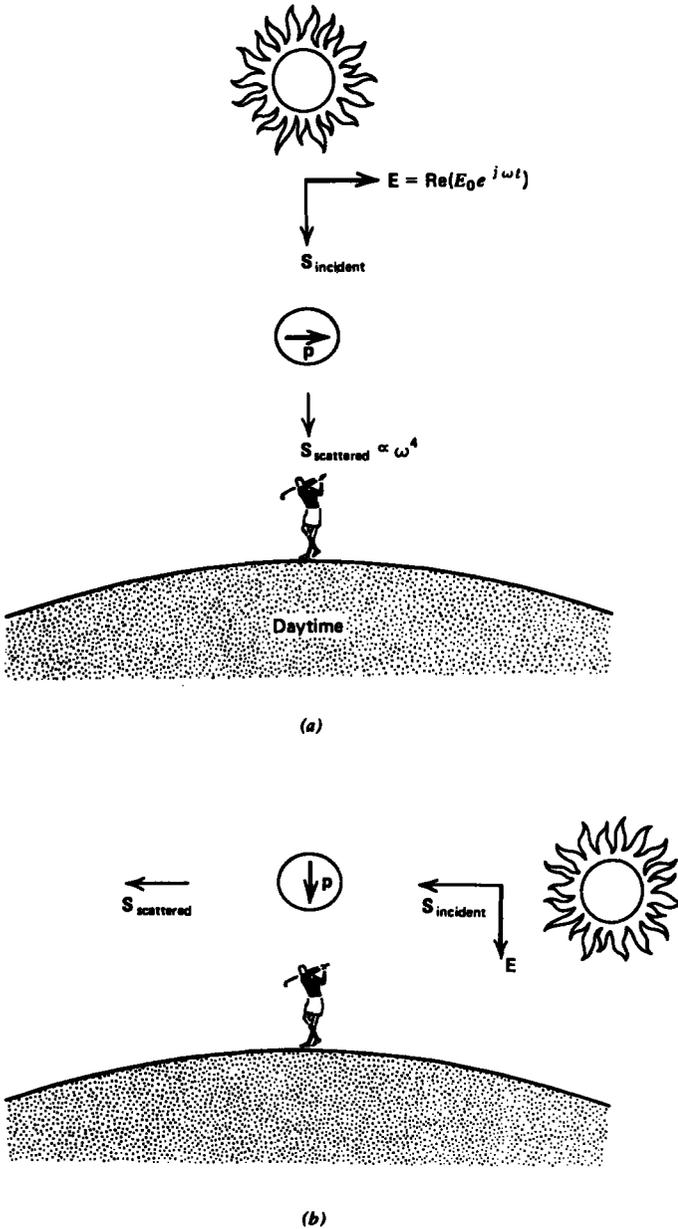


Figure 9-5 An incident electric field polarizes dipoles that then re-radiate their energy primarily perpendicular to the polarizing electric field. The time-average scattered power increases with the fourth power of frequency so shorter wavelengths of light are scattered more than longer wavelengths. (a) During the daytime an earth observer sees more of the blue scattered light so the sky looks blue (short wavelengths). (b) Near sunset the light reaching the observer lacks blue so the sky appears reddish (long wavelength).

positive point nucleus, Newton's law for the charged cloud with mass m is:

$$\frac{d^2x}{dt^2} + \omega_0^2 x = \text{Re} \left(\frac{QE_0}{m} e^{j\omega t} \right), \quad \omega_0^2 = \frac{Q^2}{4\pi\epsilon m R_0^3} \quad (40)$$

The resulting dipole moment is then

$$\hat{p} = Q\hat{x} = \frac{Q^2 E_0/m}{\omega_0^2 - \omega^2} \quad (41)$$

where we neglect damping effects. This dipole then re-radiates with solutions given in Sections 9-2-1-9-2-5 using the dipole moment of (41) ($\hat{I} dl \rightarrow j\omega\hat{p}$). The total time-average power radiated is then found from (29) as

$$\langle P \rangle = \frac{\omega^4 |\hat{p}|^2 \eta}{12\pi c^2} = \frac{\omega^4 \eta (Q^2 E_0/m)^2}{12\pi c^2 (\omega_0^2 - \omega^2)^2} \quad (42)$$

To approximately compute ω_0 , we use the approximate radius of the electron found in Section 3-8-2 by equating the energy stored in Einstein's relativistic formula relating mass to energy:

$$mc^2 = \frac{3Q^2}{20\pi\epsilon R_0} \Rightarrow R_0 = \frac{3Q^2}{20\pi\epsilon mc^2} \approx 1.69 \times 10^{-15} \text{ m} \quad (43)$$

Then from (40)

$$\omega_0 = \frac{\sqrt{5/3} 20\pi\epsilon mc^3}{3Q^2} \approx 2.3 \times 10^{23} \text{ radian/sec} \quad (44)$$

is much greater than light frequencies ($\omega \approx 10^{15}$) so that (42) becomes approximately

$$\lim_{\omega_0 \gg \omega} \langle P \rangle \approx \frac{\eta}{12\pi} \left(\frac{Q^2 E_0 \omega^2}{mc\omega_0^2} \right)^2 \quad (45)$$

This result was originally derived by Rayleigh to explain the blueness of the sky. Since the scattered power is proportional to ω^4 , shorter wavelength light dominates. However, near sunset the light is scattered parallel to the earth rather than towards it. The blue light received by an observer at the earth is diminished so that the longer wavelengths dominate and the sky appears reddish.

9-2-7 Radiation from a Point Magnetic Dipole

A closed sinusoidally varying current loop of very small size flowing in the $z = 0$ plane also generates radiating waves. Because the loop is closed, the current has no divergence so

that there is no charge and the scalar potential is zero. The vector potential phasor amplitude is then

$$\hat{\mathbf{A}}(\mathbf{r}) = \int \frac{\mu \hat{\mathbf{I}} e^{-jk r_{QP}}}{4\pi r_{QP}} dl \quad (46)$$

We assume the dipole to be much smaller than a wavelength, $k(r_{QP} - r) \ll 1$, so that the exponential factor in (46) can be linearized to

$$\lim_{k(r_{QP}-r) \ll 1} e^{-jk r_{QP}} = e^{-jkr} e^{-jk(r_{QP}-r)} \approx e^{-jkr} [1 - jk(r_{QP} - r)] \quad (47)$$

Then (46) reduces to

$$\begin{aligned} \hat{\mathbf{A}}(\mathbf{r}) &= \int \frac{\mu \hat{\mathbf{I}}}{4\pi} e^{-jkr} \left(\frac{1 + jkr}{r_{QP}} - jk \right) dl \\ &= e^{-jkr} \int \frac{\mu \hat{\mathbf{I}}}{4\pi} \left(\frac{1 + jkr}{r_{QP}} - jk \right) dl \\ &= \frac{\mu}{4\pi} e^{-jkr} \left((1 + jkr) \int \frac{\hat{\mathbf{I}} dl}{r_{QP}} - jk \int \hat{\mathbf{I}} dl \right) \end{aligned} \quad (48)$$

where all terms that depend on r can be taken outside the integrals because r is independent of dl . The second integral is zero because the vector current has constant magnitude and flows in a closed loop so that its average direction integrated over the loop is zero. This is most easily seen with a rectangular loop where opposite sides of the loop contribute equal magnitude but opposite signs to the integral, which thus sums to zero. If the loop is circular with radius a ,

$$\hat{\mathbf{I}} dl = \hat{\mathbf{I}} \mathbf{i}_\phi a d\phi \Rightarrow \int_0^{2\pi} \mathbf{i}_\phi d\phi = \int_0^{2\pi} (-\sin \phi \mathbf{i}_x + \cos \phi \mathbf{i}_y) d\phi = 0 \quad (49)$$

the integral is again zero as the average value of the unit vector \mathbf{i}_ϕ around the loop is zero.

The remaining integral is the same as for quasi-statics except that it is multiplied by the factor $(1 + jkr) e^{-jkr}$. Using the results of Section 5-5-1, the quasi-static vector potential is also multiplied by this quantity:

$$\hat{\mathbf{A}} = \frac{\mu \hat{m}}{4\pi r^2} \sin \theta (1 + jkr) e^{-jkr} \mathbf{i}_\phi, \quad \hat{m} = \hat{I} dS \quad (50)$$

The electric and magnetic fields are then

$$\begin{aligned} \hat{\mathbf{H}} &= \frac{1}{\mu} \nabla \times \hat{\mathbf{A}} = -\frac{\hat{m}}{4\pi} jk^3 e^{-jkr} \left\{ \mathbf{i}_r \left[2 \cos \theta \left(\frac{1}{(jkr)^2} + \frac{1}{(jkr)^3} \right) \right] \right. \\ &\quad \left. + \mathbf{i}_\theta \left[\sin \theta \left(\frac{1}{jkr} + \frac{1}{(jkr)^2} + \frac{1}{(jkr)^3} \right) \right] \right\} \quad (51) \\ \hat{\mathbf{E}} &= \frac{1}{j\omega\epsilon} \nabla \times \hat{\mathbf{H}} = \frac{\hat{m}jk^3}{4\pi} \eta e^{-jkr} \sin \theta \left(\frac{1}{(jkr)} + \frac{1}{(jkr)^2} \right) \mathbf{i}_\phi \end{aligned}$$

The magnetic dipole field solutions are the dual to those of the electric dipole where the electric and magnetic fields reverse roles if we replace the electric dipole moment with the magnetic dipole moment:

$$\frac{\mathbf{p}}{\epsilon} = \frac{q \, d\mathbf{l}}{\epsilon} = \frac{\mathbf{I} \, dl}{j\omega\epsilon} \rightarrow \mathbf{m} \quad (52)$$

9-3 POINT DIPOLE ARRAYS

The power density for a point electric dipole varies with the broad angular distribution $\sin^2 \theta$. Often it is desired that the power pattern be highly directive with certain angles carrying most of the power with negligible power density at other angles. It is also necessary that the directions for maximum power flow be controllable with no mechanical motion of the antenna. These requirements can be met by using more dipoles in a periodic array.

9-3-1 A Simple Two Element Array

To illustrate the basic principles of antenna arrays we consider the two element electric dipole array shown in Figure 9-6. We assume each element carries uniform currents \hat{I}_1 and \hat{I}_2 and has lengths dl_1 and dl_2 , respectively. The elements are a distance $2a$ apart. The fields at any point P are given by the superposition of fields due to each dipole alone. Since we are only interested in the far field radiation pattern where $\theta_1 \approx \theta_2 \approx \theta$, we use the solutions of Eq. (16) in Section 9-2-3 to write:

$$\hat{E}_\theta = \eta \hat{H}_\phi = \frac{\hat{E}_1 \sin \theta e^{-jkr_1}}{jkr_1} + \frac{\hat{E}_2 \sin \theta e^{-jkr_2}}{jkr_2} \quad (1)$$

where

$$\hat{E}_1 = -\frac{\hat{I}_1 \, dl_1 \, k^2}{4\pi} \eta, \quad \hat{E}_2 = -\frac{\hat{I}_2 \, dl_2 \, k^2}{4\pi} \eta$$

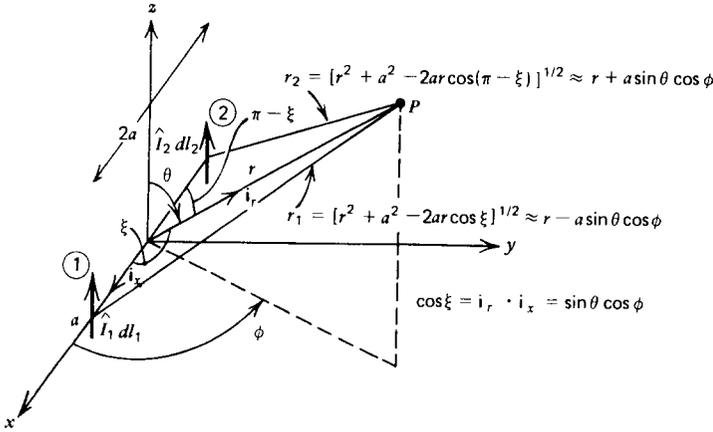


Figure 9-6 The field at any point P due to two-point dipoles is just the sum of the fields due to each dipole alone taking into account the difference in distances to each dipole.

Remember, we can superpose the fields but we cannot superpose the power flows.

From the law of cosines the distances r_1 and r_2 are related as

$$\begin{aligned} r_2 &= [r^2 + a^2 - 2ar \cos(\pi - \xi)]^{1/2} \approx r + a \sin \theta \cos \phi \\ r_1 &= [r^2 + a^2 - 2ar \cos \xi]^{1/2} \approx r - a \sin \theta \cos \phi \end{aligned} \quad (2)$$

where ξ is the angle between the unit radial vector \mathbf{i}_r and the x axis:

$$\cos \xi = \mathbf{i}_r \cdot \mathbf{i}_x = \sin \theta \cos \phi$$

Since we are interested in the far field pattern, we linearize (2) to

$$\lim_{r \gg a} \begin{cases} r_2 \approx r \left[1 + \frac{1}{2} \left(\frac{a^2}{r^2} + \frac{2a}{r} \sin \theta \cos \phi \right) \right] \approx r + a \sin \theta \cos \phi \\ r_1 \approx r \left[1 + \frac{1}{2} \left(\frac{a^2}{r^2} - \frac{2a}{r} \sin \theta \cos \phi \right) \right] \approx r - a \sin \theta \cos \phi \end{cases} \quad (3)$$

In this far field limit, the correction terms have little effect in the denominators of (1) but can have significant effect in the exponential phase factors if a is comparable to a wavelength so that ka is near or greater than unity. In this spirit we include the first-order correction terms of (3) in the phase

factors of (1), but not anywhere else, so that (1) is rewritten as

$$\hat{E}_\theta = \eta \hat{H}_\phi$$

$$= \underbrace{\frac{jk\eta}{4\pi r} \sin \theta e^{-jkr}}_{\text{element factor}} \underbrace{(\hat{I}_1 dl_1 e^{jka \sin \theta \cos \phi} + \hat{I}_2 dl_2 e^{-jka \sin \theta \cos \phi})}_{\text{array factor}} \quad (4)$$

The first factor is called the element factor because it is the radiation field per unit current element ($\hat{I} dl$) due to a single dipole at the origin. The second factor is called the array factor because it only depends on the geometry and excitations (magnitude and phase) of each dipole element in the array.

To examine (4) in greater detail, we assume the two dipoles are identical in length and that the currents have the same magnitude but can differ in phase χ :

$$dl_1 = dl_2 \equiv dl$$

$$\hat{I}_1 = \hat{I}, \quad \hat{I}_2 = \hat{I} e^{j\chi} \Rightarrow \hat{E}_1 = \hat{E}_0, \quad \hat{E}_2 = \hat{E}_0 e^{j\chi} \quad (5)$$

so that (4) can be written as

$$\hat{E}_\theta = \eta \hat{H}_\phi = \frac{2\hat{E}_0^*}{jkr} e^{-jkr} \sin \theta e^{j\chi/2} \cos \left(ka \sin \theta \cos \phi - \frac{\chi}{2} \right) \quad (6)$$

Now the far fields also depend on ϕ . In particular, we focus attention on the $\theta = \pi/2$ plane. Then the power flow,

$$\lim_{\theta=\pi/2} \langle S_r \rangle = \frac{1}{2\eta} |\hat{E}_\theta|^2 = \frac{2|\hat{E}_0|^2}{\eta(kr)^2} \cos^2 \left(ka \cos \phi - \frac{\chi}{2} \right) \quad (7)$$

depends strongly on the dipole spacing $2a$ and current phase difference χ .

(a) Broadside Array

Consider the case where the currents are in phase ($\chi = 0$) but the dipole spacing is a half wavelength ($2a = \lambda/2$). Then, as illustrated by the radiation pattern in Figure 9-7a, the field strengths cancel along the x axis while they add along the y axis. This is because along the y axis $r_1 = r_2$, so the fields due to each dipole add, while along the x axis the distances differ by a half wavelength so that the dipole fields cancel. Wherever the array factor phase ($ka \cos \phi - \chi/2$) is an integer multiple of π , the power density is maximum, while wherever it is an odd integer multiple of $\pi/2$, the power density is zero. Because this radiation pattern is maximum in the direction perpendicular to the array, it is called a broadside pattern.

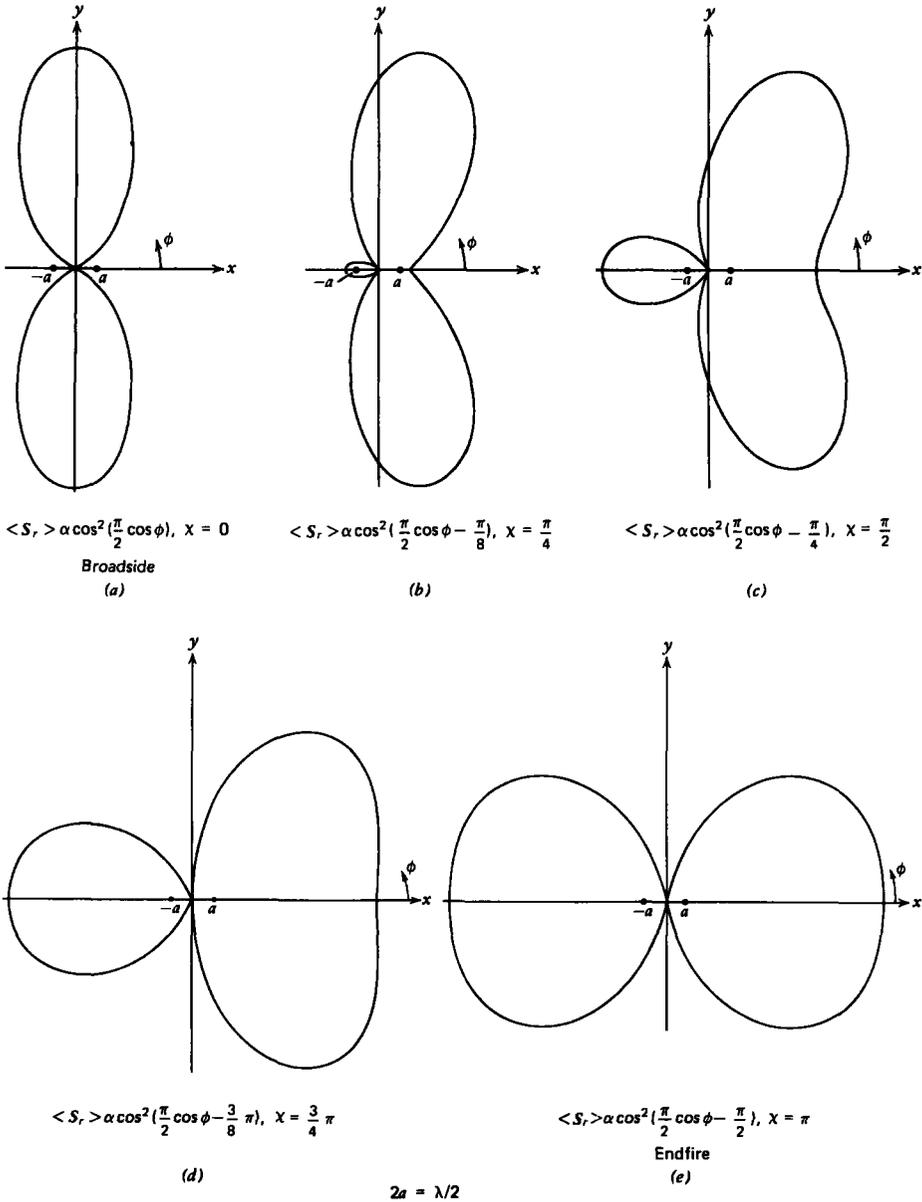


Figure 9-7 The power radiation pattern due to two-point dipoles depends strongly on the dipole spacing and current phases. With a half wavelength dipole spacing ($2a = \lambda/2$), the radiation pattern is drawn for various values of current phase difference in the $\theta = \pi/2$ plane. The broadside array in (a) with the currents in phase ($\chi = 0$) has the power lobe in the direction perpendicular to the array while the end-fire array in (e) has out-of-phase currents ($\chi = \pi$) with the power lobe in the direction along the array.

(b) End-fire Array

If, however, for the same half wavelength spacing the currents are out of phase ($\chi = \pi$), the fields add along the x axis but cancel along the y axis. Here, even though the path lengths along the y axis are the same for each dipole, because the currents are out of phase the fields cancel. Along the x axis the extra π phase because of the half wavelength path difference is just canceled by the current phase difference of π so that the fields due to each dipole add. The radiation pattern is called end-fire because the power is maximum in the direction along the array, as shown in Figure 9-7e.

(c) Arbitrary Current Phase

For arbitrary current phase angles and dipole spacings, a great variety of radiation patterns can be obtained, as illustrated by the sequences in Figures 9-7 and 9-8. More power lobes appear as the dipole spacing is increased.

9-3-2 An N Dipole Array

If we have $(2N + 1)$ equally spaced dipoles, as shown in Figure 9-9, the n th dipole's distance to the far field point is approximately,

$$\lim_{r \gg |na|} r_n \approx r - na \sin \theta \cos \phi \quad (8)$$

so that the array factor of (4) generalizes to

$$AF = \sum_{-N}^{+N} \hat{I}_n dl_n e^{jkn a \sin \theta \cos \phi} \quad (9)$$

where for symmetry we assume that there are as many dipoles to the left (negative n) as to the right (positive n) of the z axis, including one at the origin ($n = 0$). In the event that a dipole is not present at a given location, we simply let its current be zero. The array factor can be varied by changing the current magnitude or phase in the dipoles. For simplicity here, we assume that all dipoles have the same length dl , the same current magnitude I_0 , and differ in phase from its neighbors by a constant angle χ_0 so that

$$\hat{I}_n = I_0 e^{-jn\chi_0}, \quad -N \leq n \leq N \quad (10)$$

and (9) becomes

$$AF = I_0 dl \sum_{-N}^{+N} e^{jm(ka \sin \theta \cos \phi - \chi_0)} \quad (11)$$

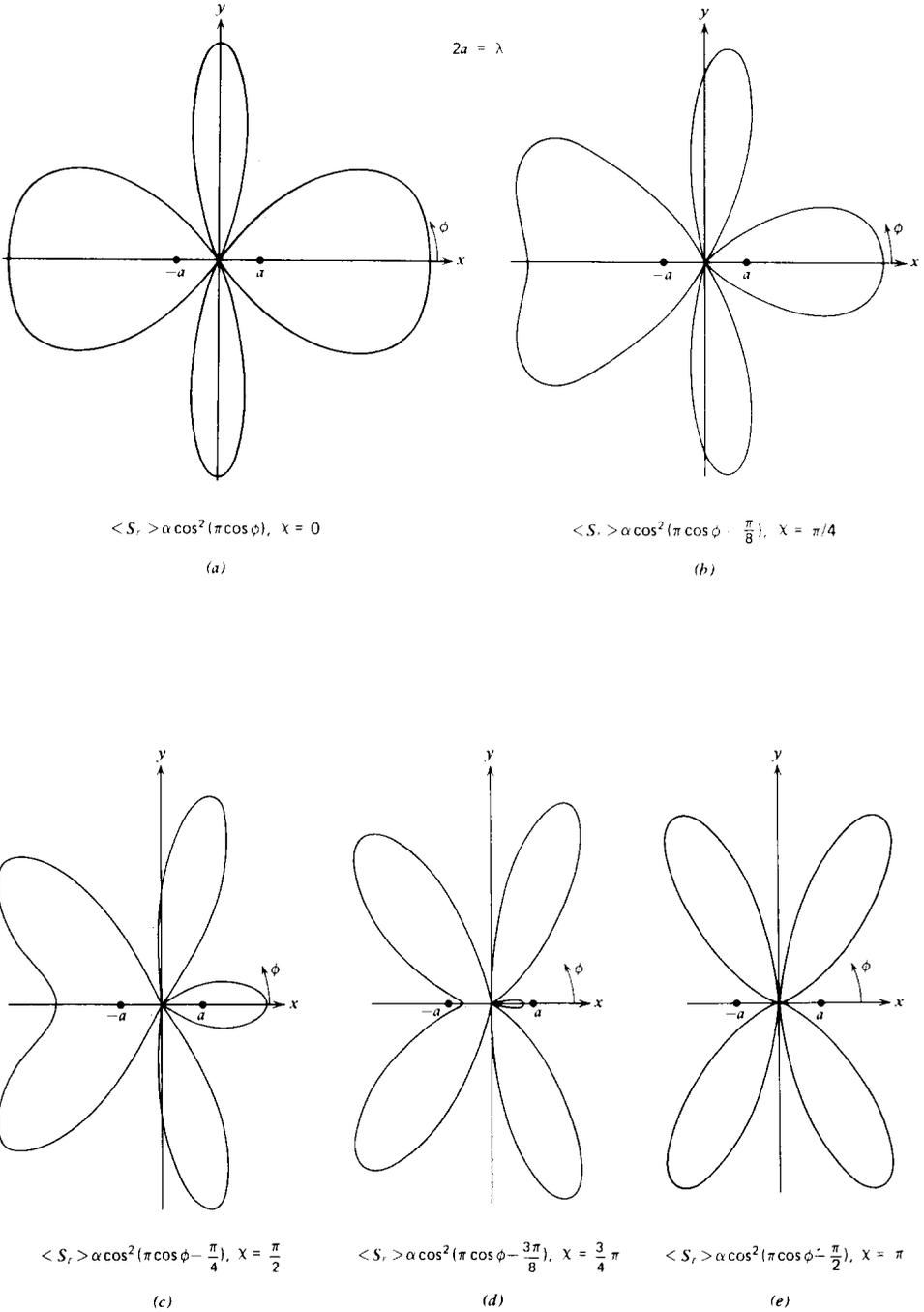


Figure 9-8 With a full wavelength dipole spacing ($2a = \lambda$) there are four main power lobes.

Defining the parameter

$$\beta = e^{j(ka \sin \theta \cos \phi - \chi_0)} \quad (12)$$

the geometric series in (11) can be written as

$$S = \sum_{-N}^{+N} \beta^n = \beta^{-N} + \beta^{-N+1} + \cdots + \beta^{-2} + \beta^{-1} + 1 + \beta + \beta^2 + \cdots + \beta^{N-1} + \beta^N \quad (13)$$

If we multiply this series by β and subtract from (13), we have

$$S(1 - \beta) = \beta^{-N} - \beta^{N+1} \quad (14)$$

which allows us to write the series sum in closed form as

$$\begin{aligned} S &= \frac{\beta^{-N} - \beta^{N+1}}{1 - \beta} = \frac{\beta^{-(N+1/2)} - \beta^{(N+1/2)}}{\beta^{-1/2} - \beta^{1/2}} \\ &= \frac{\sin [(N + \frac{1}{2})(ka \sin \theta \cos \phi - \chi_0)]}{\sin [\frac{1}{2}(ka \sin \theta \cos \phi - \chi_0)]} \end{aligned} \quad (15)$$

In particular, we again focus on the solution in the $\theta = \pi/2$ plane so that the array factor is

$$AF = I_0 dl \frac{\sin [(N + \frac{1}{2})(ka \cos \phi - \chi_0)]}{\sin [\frac{1}{2}(ka \cos \phi - \chi_0)]} \quad (16)$$

The radiation pattern is proportional to the square of the array factor. Maxima occur where

$$ka \cos \phi - \chi_0 = 2n\pi \quad n = 0, 1, 2, \dots \quad (17)$$

The principal maximum is for $n = 0$ as illustrated in Figure 9-10 for various values of ka and χ_0 . The larger the number of dipoles N , the narrower the principal maximum with smaller amplitude side lobes. This allows for a highly directive beam at angle ϕ controlled by the incremental current phase angle χ_0 , so that $\cos \phi = \chi_0/ka$, which allows for electronic beam steering by simply changing χ_0 .

9-4 LONG DIPOLE ANTENNAS

The radiated power, proportional to $(dl/\lambda)^2$, is small for point dipole antennas where the dipole's length dl is much less than the wavelength λ . More power can be radiated if the length of the antenna is increased. Then however, the fields due to each section of the antenna may not add constructively.

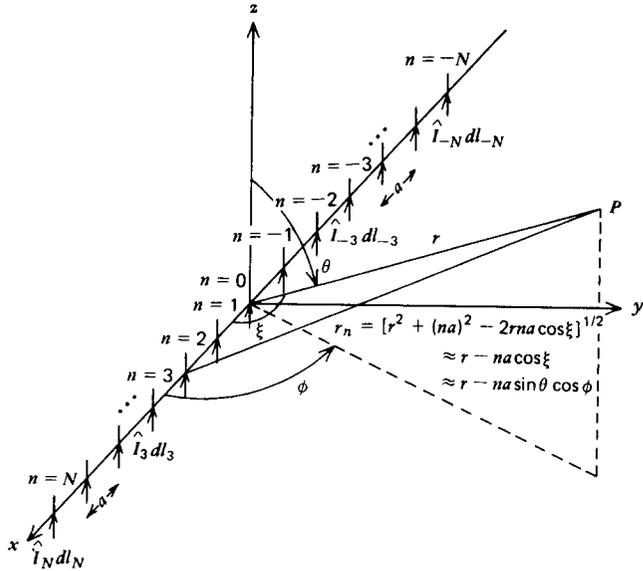


Figure 9-9 A linear point dipole array with $2N + 1$ equally spaced dipoles.

9-4-1 Far Field Solution

Consider the long dipole antenna in Figure 9-11 carrying a current $\hat{I}(z)$. For simplicity we restrict ourselves to the far field pattern where $r \gg L$. Then, as we found for dipole arrays, the differences in radial distance for each incremental current element of length dz are only important in the exponential phase factors and not in the $1/r$ dependences.

From Section 9-2-3, the incremental current element at position z generates a far electric field:

$$d\hat{E}_\theta = \eta d\hat{H}_\phi = \frac{jk\eta}{4\pi} \frac{\hat{I}(z) dz}{r} \sin \theta e^{-jk(r-z \cos \theta)} \quad (1)$$

where we again assume that in the far field the angle θ is the same for all incremental current elements.

The total far electric field due to the entire current distribution is obtained by integration over all current elements:

$$\hat{E}_\theta = \eta \hat{H}_\phi = \frac{jk\eta}{4\pi r} \sin \theta e^{-jkr} \int_{-L/2}^{+L/2} \hat{I}(z) e^{jkz \cos \theta} dz \quad (2)$$

If the current distribution is known, the integral in (2) can be directly evaluated. The practical problem is difficult because the current distribution along the antenna is determined by the near fields through the boundary conditions.

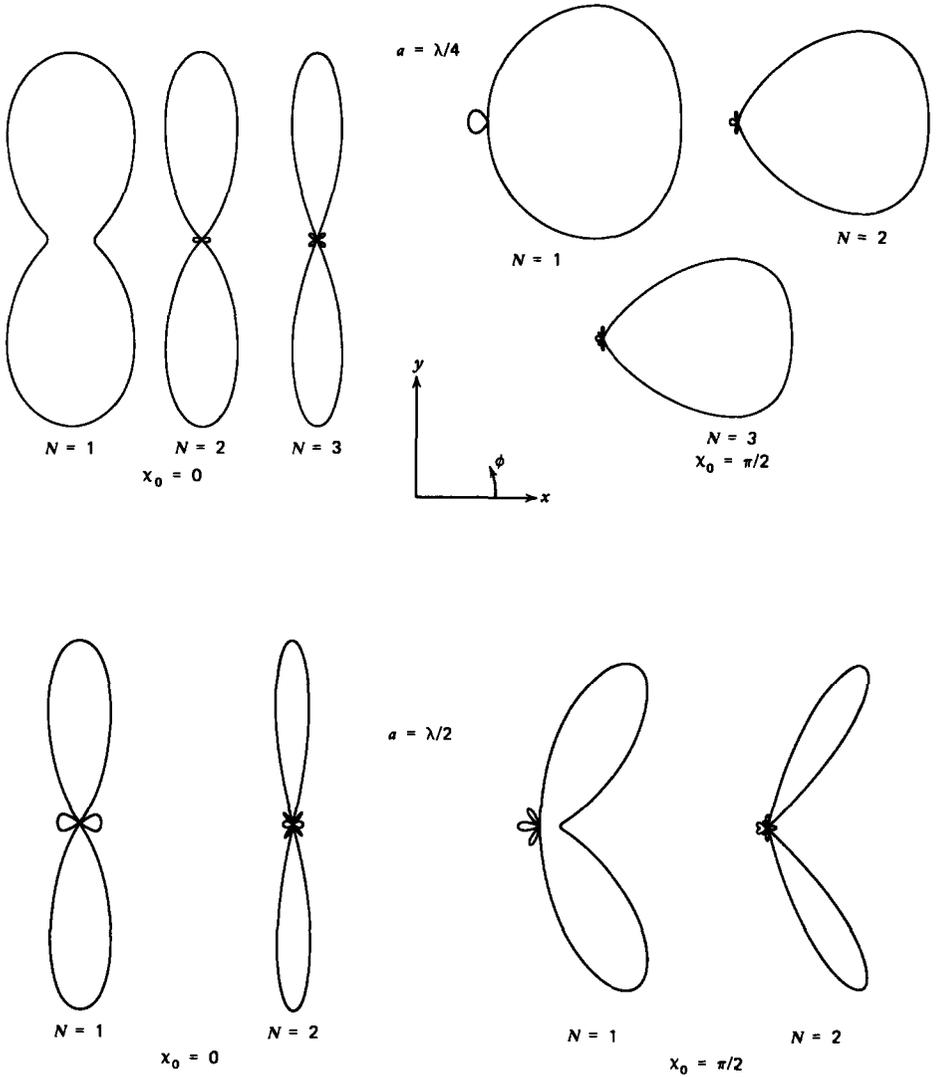


Figure 9-10 The radiation pattern for an N dipole linear array for various values of N , dipole spacing $2a$, and relative current phase χ_0 in the $\theta = \pi/2$ plane.

Since the fields and currents are coupled, an exact solution is impossible no matter how simple the antenna geometry. In practice, one guesses a current distribution and calculates the resultant (near and far) fields. If all boundary conditions along the antenna are satisfied, then the solution has been found. Unfortunately, this never happens with the first guess. Thus based on the field solution obtained from the originally guessed current, a corrected current distribution is used and

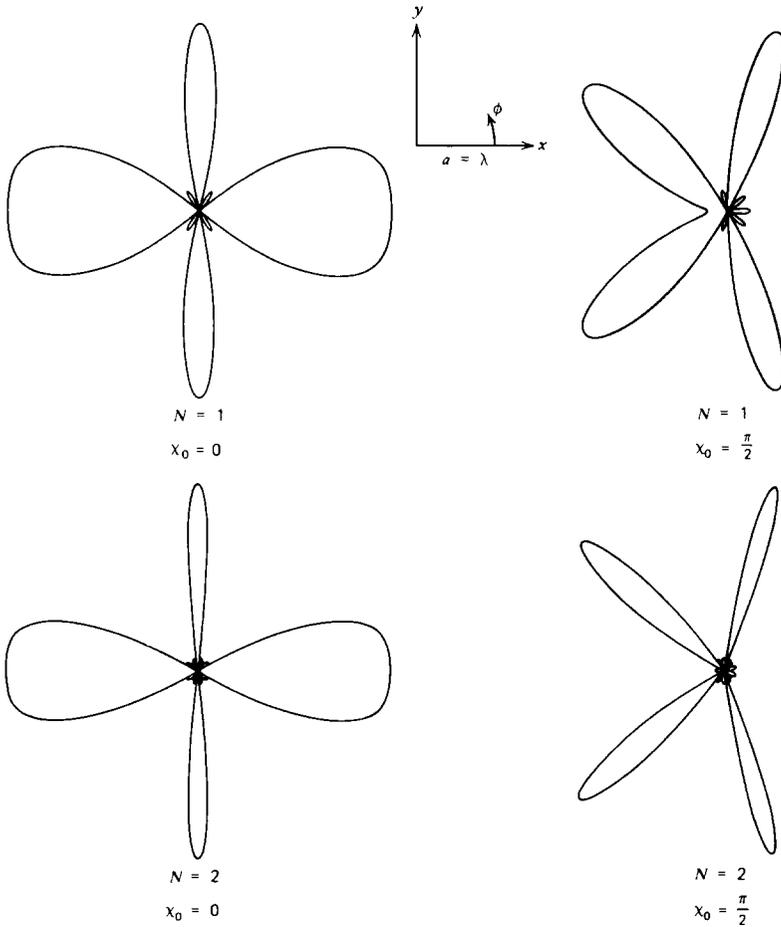


Figure 9-10

the resulting fields are again calculated. This procedure is numerically iterated until convergence is obtained with self-consistent fields and currents.

9-4-2 Uniform Current

A particularly simple case is when $\hat{I}(z) = \hat{I}_0$ is a constant. Then (2) becomes:

$$\begin{aligned}
 \hat{E}_\theta &= \eta \hat{H}_\phi = \frac{jk\eta}{4\pi r} \sin \theta e^{-jkr} \hat{I}_0 \int_{-L/2}^{+L/2} e^{jkz \cos \theta} dz \\
 &= \frac{jk\eta}{4\pi r} \sin \theta e^{-jkr} \hat{I}_0 \left. \frac{e^{jkz \cos \theta}}{jk \cos \theta} \right|_{-L/2}^{+L/2} \\
 &= \frac{\hat{I}_0 \eta}{4\pi r} \tan \theta e^{-jkr} \left[2j \sin \left(\frac{kL}{2} \cos \theta \right) \right] \quad (3)
 \end{aligned}$$

The time-average power density is then

$$\langle S_r \rangle = \frac{1}{2\eta} |\hat{E}_\theta|^2 = \frac{|\hat{E}_0|^2 \tan^2 \theta \sin^2 [(kL/2) \cos \theta]}{2\eta(kr)^2 (kL/2)^2} \quad (4)$$

where

$$\hat{E}_0 = \frac{\hat{I}_0 L \eta k^2}{4\pi} \quad (5)$$

This power density is plotted versus angle θ in Figure 9-12 for various lengths L . The principal maximum always appears at $\theta = \pi/2$, becoming sharper as L increases. For $L > \lambda$, zero power density occurs at angles

$$\cos \theta = \frac{2n\pi}{kL} = \frac{n\lambda}{L}, \quad n = 1, 2, \dots \quad (6)$$

Secondary maxima then occur at nearby angles but at much smaller amplitudes compared to the main lobe at $\theta = \pi/2$.

9-4-3 Radiation Resistance

The total time-average radiated power is obtained by integrating (4) over all angles:

$$\begin{aligned} \langle P \rangle &= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \langle S_r \rangle r^2 \sin \theta \, d\theta \, d\phi \\ &= \frac{|\hat{E}_0|^2 \pi}{k^2 \eta (kL/2)^2} \int_{\theta=0}^{\pi} \frac{\sin^3 \theta}{\cos^2 \theta} \sin^2 \left(\frac{kL}{2} \cos \theta \right) d\theta \end{aligned} \quad (7)$$

If we introduce the change of variable,

$$v = \frac{kL}{2} \cos \theta, \quad dv = -\frac{kL}{2} \sin \theta \, d\theta \quad (8)$$

the integral of (7) becomes

$$\langle P \rangle = \frac{|\hat{E}_0|^2 \pi}{k^2 \eta (kL/2)^2} \int_{+kL/2}^{-kL/2} \left(\frac{2}{kL} \sin^2 v \, dv - \frac{kL}{2} \frac{\sin^2 v}{v^2} \right) dv \quad (9)$$

The first term is easily integrable as

$$\int \sin^2 v \, dv = \frac{1}{2}v - \frac{1}{4} \sin 2v \quad (10)$$

The second integral results in a new tabulated function $Si(x)$ called the sine integral, defined as:

$$Si(x) = \int_0^x \frac{\sin t}{t} dt \quad (11)$$

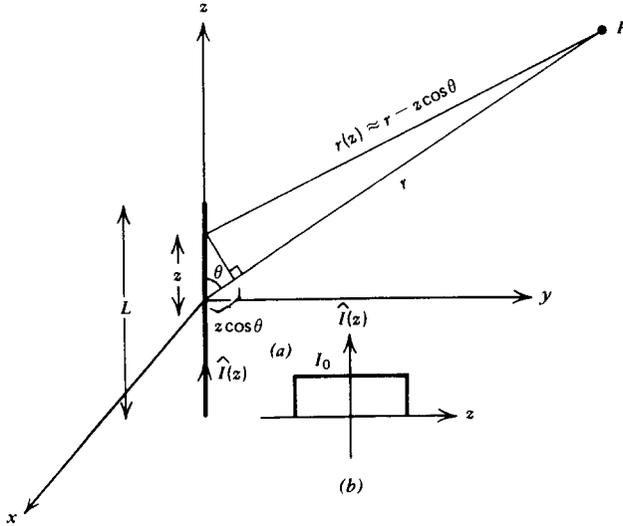


Figure 9-11 (a) For a long dipole antenna, each incremental current element at coordinate z is at a slightly different distance to any field point P . (b) The simplest case study has the current uniformly distributed over the length of the dipole.

which is plotted in Figure 9-13. Then the second integral in (9) can be expanded and integrated by parts:

$$\begin{aligned}
 \int \frac{\sin^2 v}{v^2} dv &= \int \frac{(1 - \cos 2v)}{2v^2} dv \\
 &= -\frac{1}{2v} - \int \cos 2v \frac{dv}{2v^2} \\
 &= -\frac{1}{2v} + \frac{\cos 2v}{2v} + \int \frac{\sin 2v d(2v)}{2v} \\
 &= -\frac{1}{2v} + \frac{\cos 2v}{2v} + Si(2v) \tag{12}
 \end{aligned}$$

Then evaluating the integrals of (10) and (12) in (9) at the upper and lower limits yields the time-average power as:

$$\langle P \rangle = \frac{|\hat{E}_0|^2 \pi}{k^2 \eta (kL/2)^2} \left(\frac{\sin kL}{kL} + \cos kL - 2 + kL Si(kL) \right) \tag{13}$$

where we used the fact that the sine integral is an odd function $Si(x) = -Si(x)$.

Using (5), the radiation resistance is then

$$R = \frac{2\langle P \rangle}{|\hat{I}_0|^2} = \frac{\eta}{2\pi} \left(\frac{\sin kL}{kL} + \cos kL - 2 + kL Si(kL) \right) \tag{14}$$

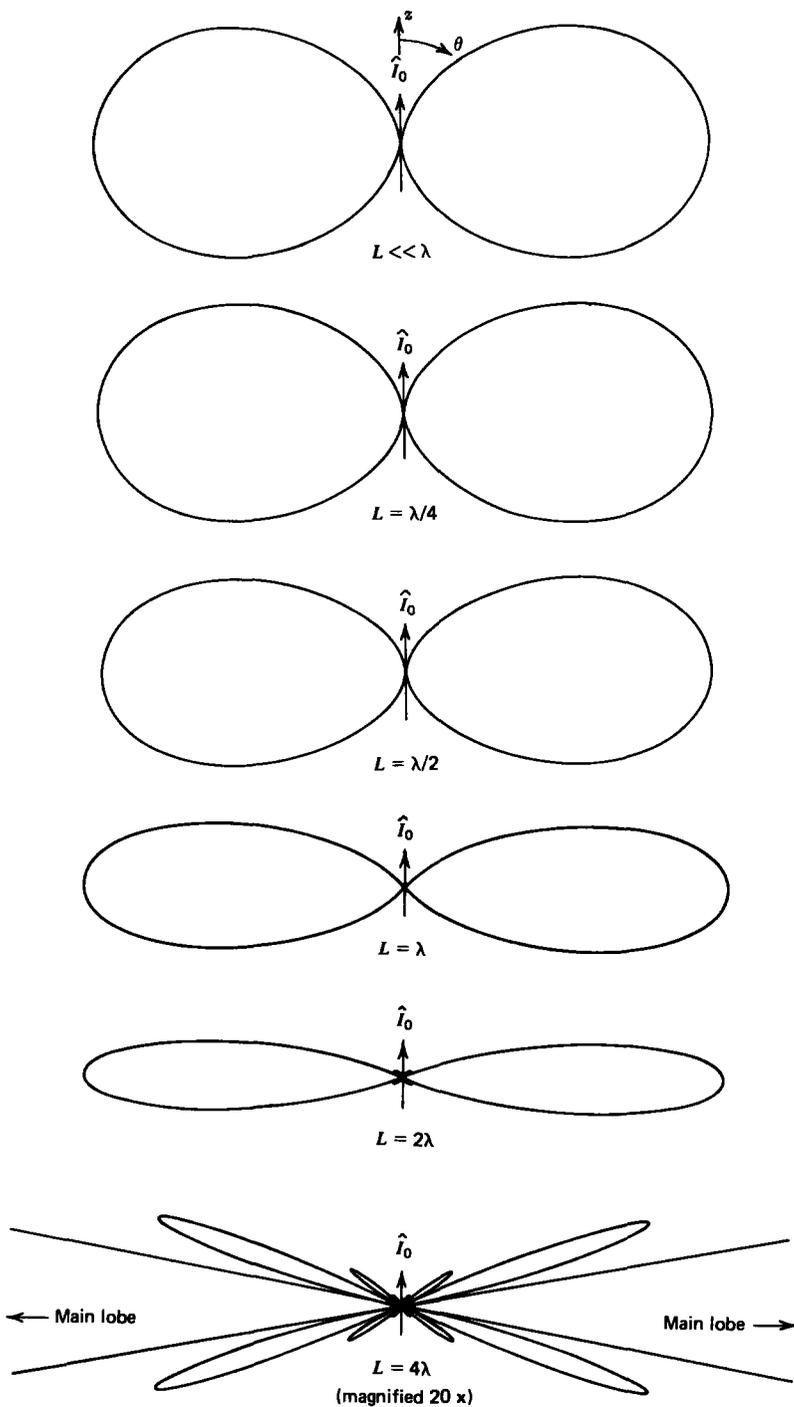


Figure 9-12 The radiation pattern for a long dipole for various values of its length.

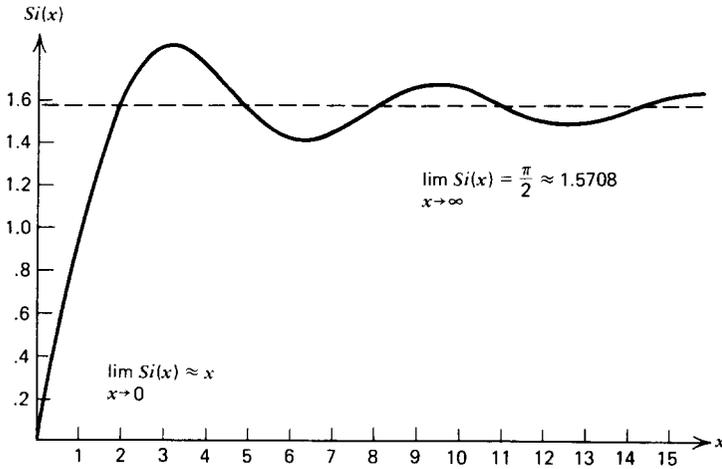


Figure 9-13 The sine integral $Si(x)$ increases linearly for small arguments and approaches $\pi/2$ for large arguments oscillating about this value for intermediate arguments.

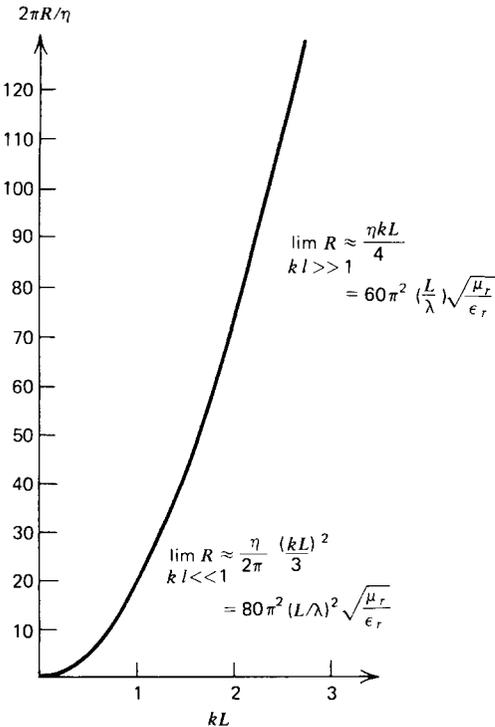


Figure 9-14 The radiation resistance for a dipole antenna carrying a uniformly distributed current increases with the square of its length when it is short ($L/\lambda \ll 1$) and only linearly with its length when it is long ($L/\lambda \gg 1$). For short lengths, the radiation resistance approximates that of a point dipole.

which is plotted versus kL in Fig. 9-14. This result can be checked in the limit as L becomes very small ($kL \ll 1$) since the radiation resistance should approach that of a point dipole given in Section 9-2-5. In this short dipole limit the bracketed terms in (14) are

$$\lim_{kL \ll 1} \left\{ \begin{array}{l} \frac{\sin kL}{kL} \approx 1 - \frac{(kL)^2}{6} \\ \cos kL \approx 1 - \frac{(kL)^2}{2} \\ kL \text{Si}(kL) \approx (kL)^2 \end{array} \right. \quad (15)$$

so that (14) reduces to

$$\lim_{kL \ll 1} R \approx \frac{\eta}{2\pi} \frac{(kL)^2}{3} = \frac{2\pi\eta}{3} \left(\frac{L}{\lambda}\right)^2 = 80\pi^2 \left(\frac{L}{\lambda}\right)^2 \sqrt{\frac{\mu_r}{\epsilon_r}} \quad (16)$$

which agrees with the results in Section 9-2-5. Note that for large dipoles ($kL \gg 1$), the sine integral term dominates with $\text{Si}(kL)$ approaching a constant value of $\pi/2$ so that

$$\lim_{kL \gg 1} R \approx \frac{\eta kL}{4} = 60 \sqrt{\frac{\mu_r}{\epsilon_r}} \pi^2 \frac{L}{\lambda} \quad (17)$$

PROBLEMS

Section 9-1

1. We wish to find the properties of waves propagating within a linear dielectric medium that also has an Ohmic conductivity σ .

- (a) What are Maxwell's equations in this medium?
- (b) Defining vector and scalar potentials, what gauge condition decouples these potentials?
- (c) A point charge at $r = 0$ varies sinusoidally with time as $Q(t) = \text{Re}(\hat{Q} e^{j\omega t})$. What is the scalar potential?
- (d) Repeat (a)–(c) for waves in a plasma medium with constitutive law

$$\frac{\partial \mathbf{J}_f}{\partial t} = \omega_p^2 \epsilon \mathbf{E}$$

2. An infinite current sheet at $z = 0$ varies as $\text{Re}[K_0 e^{j(\omega t - k_x x)} \mathbf{i}_x]$.

- (a) Find the vector and scalar potentials.
- (b) What are the electric and magnetic fields?

(c) Repeat (a) and (b) if the current is uniformly distributed over a planar slab of thickness $2a$:

$$\mathbf{J}_f = \begin{cases} J_0 e^{j(\omega t - k_z z)} \mathbf{i}_z, & -a < z < a \\ 0, & |z| > a \end{cases}$$

3. A sphere of radius R has a uniform surface charge distribution $\sigma_f = \text{Re}(\hat{\sigma}_0 e^{j\omega t})$ where the time varying surface charge is due to a purely radial conduction current.

(a) Find the scalar and vector potentials, inside and outside the sphere. (**Hint:** $r_{QP}^2 = r^2 + R^2 - 2rR \cos \theta$; $r_{QP} dr_{QP} = rR \sin \theta d\theta$.)

(b) What are the electric and magnetic fields everywhere?

Section 9.2

4. Find the effective lengths, radiation resistances and line charge distributions for each of the following current distributions valid for $|z| < dl/2$ on a point electric dipole with short length dl :

(a) $\hat{I}(z) = I_0 \cos \alpha z$

(b) $\hat{I}(z) = I_0 e^{-\alpha|z|}$

(c) $\hat{I}(z) = I_0 \cosh \alpha z$

5. What is the time-average power density, total time-average power, and radiation resistance of a point magnetic dipole?

6. A plane wave electric field $\text{Re}(\mathbf{E}_0 e^{j\omega t})$ is incident upon a perfectly conducting spherical particle of radius R that is much smaller than the wavelength.

(a) What is the induced dipole moment? (**Hint:** See Section 4-4-3.)

(b) If the small particle is, instead, a pure lossless dielectric with permittivity ϵ , what is the induced dipole moment?

(c) For both of these cases, what is the time-average scattered power?

7. A plane wave magnetic field $\text{Re}(\mathbf{H}_0 e^{j\omega t})$ is incident upon a perfectly conducting particle that is much smaller than the wavelength.

(a) What is the induced magnetic dipole moment? (**Hint:** See Section 5-7-2ii and 5-5-1.)

(b) What are the re-radiated electric and magnetic fields?

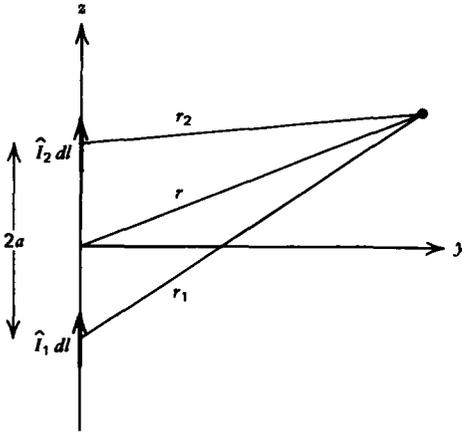
(c) What is the time-average scattered power? How does it vary with frequency?

8. (a) For the magnetic dipole, how are the magnetic field lines related to the vector potential \mathbf{A} ?

(b) What is the equation of these field lines?

Section 9.3

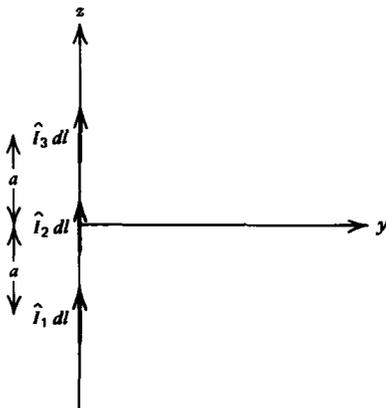
9. Two aligned dipoles $\hat{I}_1 dl$ and $\hat{I}_2 dl$ are placed along the z axis a distance $2a$ apart. The dipoles have the same length



while the currents have equal magnitudes but phase difference χ .

- What are the far electric and magnetic fields?
- What is the time-average power density?
- At what angles is the power density zero or maximum?
- For $2a = \lambda/2$, what values of χ give a broadside or end-fire array?
- Repeat (a)–(c) for $2N + 1$ equally spaced aligned dipoles along the z axis with incremental phase difference χ_0 .

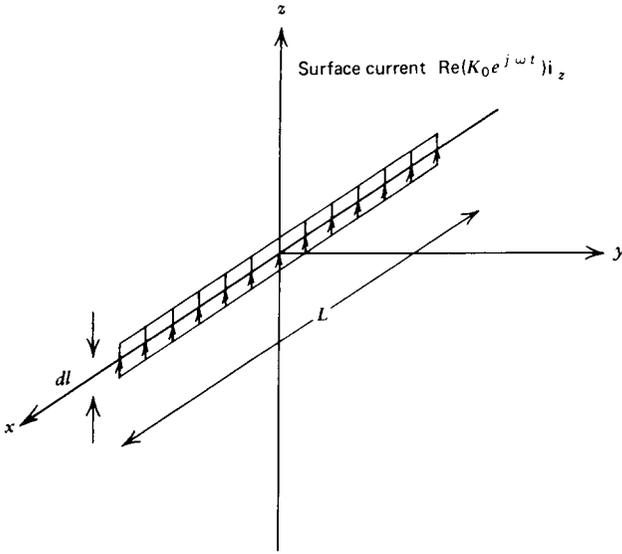
10. Three dipoles of equal length dl are placed along the z axis.



- Find the far electric and magnetic fields.
- What is the time average power density?
- For each of the following cases find the angles where the power density is zero or maximum.

- $\hat{I}_1 = \hat{I}_3 = I_0, \hat{I}_2 = 2I_0$
- $\hat{I}_1 = \hat{I}_3 = I_0, \hat{I}_2 = -2I_0$
- $\hat{I}_1 = -\hat{I}_3 = I_0, \hat{I}_2 = 2jI_0$

11. Many closely spaced point dipoles of length dl placed along the x axis driven in phase approximate a z -directed current sheet $\text{Re}(K_0 e^{j\omega t} \mathbf{i}_z)$ of length L .



- (a) Find the far fields from this current sheet.
- (b) At what angles is the power density minimum or maximum?

Section 9.4

12. Find the far fields and time-average power density for each of the following current distributions on a long dipole:

(a) $\hat{I}(z) = \begin{cases} I_0(1 - 2z/L), & 0 < z < L/2 \\ I_0(1 + 2z/L), & -L/2 < z < 0 \end{cases}$

Hint:

$$\int z e^{az} dz = \frac{e^{az}}{a^2} (az - 1)$$

(b) $\hat{I}(z) = I_0 \cos \pi z/L, \quad -L/2 < z < L/2$

Hint:

$$\int e^{az} \cos pz dz = e^{az} \frac{(a \cos pz + p \sin pz)}{(a^2 + p^2)}$$

- (c) For these cases find the radiation resistance when $kL \ll 1$.