

# Chapter 1

## Fields

1.1	Action at a Distance versus Field Theory .....	1
1.2	Scalar Fields.....	2
	Example 1.1: Half-Frozen /Half-Baked Planet.....	4
1.2.1	Representations of a Scalar Field.....	4
1.3	Vector Fields.....	6
1.4	Fluid Flow .....	7
	<b>Animation 1.1:</b> Sources and Sinks.....	7
	<b>Animation 1.2:</b> Circulations .....	9
1.4.1	Relationship Between Fluid Fields and Electromagnetic Fields .....	11
1.5	Gravitational Field.....	11
1.6	Electric Fields .....	13
1.7	Magnetic Field.....	14
1.8	Representations of a Vector Field.....	15
1.8.1	Vector Field Representation .....	15
1.8.2	Field Line Representation.....	16
1.8.3	Grass Seeds and Iron Filings Representations .....	17
1.8.4	Motion of Electric and Magnetic Field Lines .....	18
1.9	Summary .....	18
1.10	Solved Problems .....	19
1.10.1	Vector Fields.....	19
1.10.2	Scalar Fields.....	21
1.11	Additional Problems .....	22
1.11.1	Plotting Vector Fields .....	22
1.11.2	Position Vector in Spherical Coordinates .....	22
1.11.3	Electric Field.....	22
1.11.4	An Object Moving in a Circle.....	22
1.11.5	Vector Fields.....	23
1.11.6	Object Moving in Two Dimensions.....	23
1.11.7	Law of Cosines .....	24
1.11.8	Field Lines .....	24

# Fields

## 1.1 Action at a Distance versus Field Theory

“... In order therefore to appreciate the requirements of the science [of electromagnetism], the student must make himself familiar with a considerable body of most intricate mathematics, the mere retention of which in the memory materially interferes with further progress ...”

James Clerk Maxwell [1855]

Classical electromagnetic field theory emerged in more or less complete form in 1873 in James Clerk Maxwell's *A Treatise on Electricity and Magnetism*. Maxwell based his theory in large part on the intuitive insights of Michael Faraday. The wide acceptance of Maxwell's theory has caused a fundamental shift in our understanding of physical reality. In this theory, electromagnetic fields are the mediators of the interaction between material objects. This view differs radically from the older “action at a distance” view that preceded field theory.

What is “action at a distance?” It is a worldview in which the interaction of two material objects requires no mechanism other than the objects themselves and the empty space between them. That is, two objects exert a force on each other simply because they are present. Any mutual force between them (for example, gravitational attraction or electric repulsion) is instantaneously transmitted from one object to the other through empty space. There is no need to take into account any method or agent of transmission of that force, or any finite speed for the propagation of that agent of transmission. This is known as “action at a distance” because objects exert forces on one another (“action”) with nothing but empty space (“distance”) between them. No other agent or mechanism is needed.

Many natural philosophers objected to the “action at a distance” model because in our everyday experience, forces are exerted by one object on another only when the objects are in direct contact. In the field theory view, this is always true in some sense. That is, objects that are not in direct contact (objects separated by apparently empty space) must exert a force on one another *through the presence of an intervening medium or mechanism existing in the space between the objects*.

The force between the two objects is transmitted by direct “contact” from the first object to an intervening mechanism immediately surrounding that object, and then from one element of space to a neighboring element, in a continuous manner, until the force is transmitted to the region of space contiguous to the second object, and thus ultimately to the second object itself.

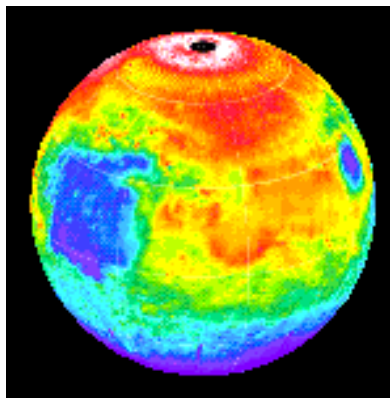
Although the two objects are not in direct contact with one another, they are in direct contact with a medium or mechanism that exists between them. The force between the objects is transmitted (at a finite speed) by stresses induced in the intervening space by the presence of the objects. The “field theory” view thus avoids the concept of “action at a distance” and replaces it by the concept of “action by continuous contact.” The “contact” is provided by a stress, or “field,” induced in the space between the objects by their presence.

This is the essence of field theory, and is the foundation of all modern approaches to understanding the world around us. Classical electromagnetism was the first field theory. It involves many concepts that are mathematically complex. As a result, even now it is difficult to appreciate. In this first chapter of your introduction to field theory, we discuss what a *field* is, and how we represent fields. We begin with scalar fields.

## 1.2 Scalar Fields

A *scalar field* is a function that gives us a single value of some variable for every point in space. As an example, the image in Figure 1.2.1 shows the nighttime temperatures measured by the Thermal Emission Spectrometer instrument on the Mars Global Surveyor (MGS). The data were acquired during the first 500 orbits of the MGS mapping mission. The coldest temperatures, shown in purple, are  $-120^{\circ}\text{C}$  while the warmest, shown in white, are  $-65^{\circ}\text{C}$ .

The view is centered on Isidis Planitia ( $15\text{N}$ ,  $270\text{W}$ ), which is covered with warm material, indicating a sandy and rocky surface. The small, cold (blue) circular region to the right is the area of the Elysium volcanoes, which are covered in dust that cools off rapidly at night. At this season the north polar region is in full sunlight and is relatively warm at night. It is winter in the southern hemisphere and the temperatures are extremely low.

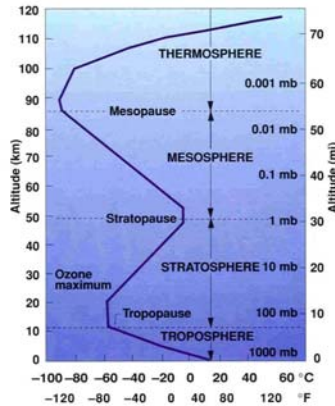


**Figure 1.2.1** Nighttime temperature map for Mars

The various colors on the map represent the surface temperature. This map, however, is limited to representing only the temperature on a two-dimensional surface and thus, it does not show how temperature varies as a function of altitude. In principal, a scalar

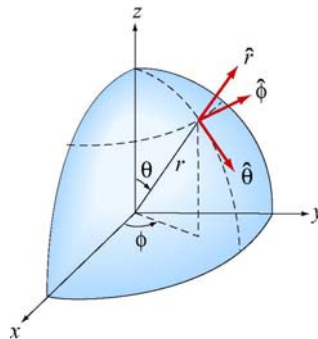
field provides values not only on a two-dimensional surface in space but for every point in space.

Figure 1.2.2 illustrates the variation of temperature as a function of height above the surface of the Earth, which is a third dimension which complements the two dimensions shown in Figure 1.2.1.



**Figure 1.2.2** Atmospheric temperature variation as a function of altitude above the Earth's surface

How do we represent three-dimensional scalar fields? In principle, one could create a three-dimensional atmospheric volume element and color it to represent the temperature variation.



**Figure 1.2.3** Spherical coordinates

Another way is to simply represent the temperature variation by a mathematical function. For the Earth we shall use spherical coordinates  $(r, \theta, \phi)$  shown in Figure 1.2.3 with the origin chosen to coincide with the center of the Earth. The temperature at any point is characterized by a function  $T(r, \theta, \phi)$ . In other words, the value of this function at the point with coordinates  $(r, \theta, \phi)$  is a temperature with given units. The temperature function  $T(r, \theta, \phi)$  is an example of a “scalar field.” The term “scalar” implies that temperature at any point is a number rather than a vector (a vector has both magnitude and direction).

### Example 1.1: Half-Frozen /Half-Baked Planet

As an example of a scalar field, consider a planet with an atmosphere that rotates with the same angular frequency about its axis as the planet orbits about a nearby star, *i.e.*, one hemisphere always faces the star. Let  $R$  denote the radius of the planet. Use spherical coordinates  $(r, \theta, \phi)$  with the origin at the center of the planet, and choose  $\phi = \pi/2$  for the center of the hemisphere facing the star. A simplistic model for the temperature variation at any point is given by

$$T(r, \theta, \phi) = [T_0 + T_1 \sin^2 \theta + T_2(1 + \sin \phi)] e^{-\alpha(r-R)} \quad (1.2.1)$$

where  $T_0, T_1, T_2$ , and  $\alpha$  are constants. The dependence on the variable  $r$  in the term  $e^{-\alpha(r-R)}$  indicates that the temperature decreases exponentially as we move radially away from the surface of the planet. The dependence on the variable  $\theta$  in the term  $\sin^2 \theta$  implies that the temperature decreases as we move toward the poles. Finally, the  $\phi$  dependence in the term  $(1 + \sin \phi)$  indicates that the temperature decreases as we move away from the center of the hemisphere facing the star.

A scalar field can also be used to describe other physical quantities such as the atmospheric pressure. However, a single number (magnitude) at every point in space is not sufficient to characterize quantities such as the wind velocity since a direction at every point in space is needed as well.

#### 1.2.1 Representations of a Scalar Field

A field, as stated earlier, is a function that has a different value at every point in space. A scalar field is a field for which there is a single number associated with every point in space. We have seen that the temperature of the Earth's atmosphere at the surface is an example of a scalar field. Another example is

$$\phi(x, y, z) = \frac{1}{\sqrt{x^2 + (y+d)^2 + z^2}} - \frac{1/3}{\sqrt{x^2 + (y-d)^2 + z^2}} \quad (1.2.2)$$

This expression defines the value of the scalar function  $\phi$  at every point  $(x, y, z)$  in space. How do visually represent a scalar field defined by an equation such as Eq. (1.2.2)? Below we discuss three possible representations.

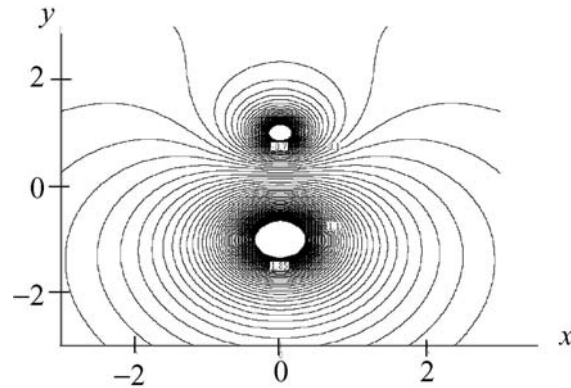
#### 1. Contour Maps

One way is to fix one of our independent variables ( $z$ , for example) and then show a contour map for the two remaining dimensions, in which the curves represent lines of constant values of the function  $\phi$ . A series of these maps for various (fixed) values of  $z$

then will give a feel for the properties of the scalar function. We show such a contour map in the  $xy$ -plane at  $z = 0$  for Eq. (1.2.2), namely,

$$\phi(x, y, 0) = \frac{1}{\sqrt{x^2 + (y+d)^2}} - \frac{1/3}{\sqrt{x^2 + (y-d)^2}} \quad (1.2.3)$$

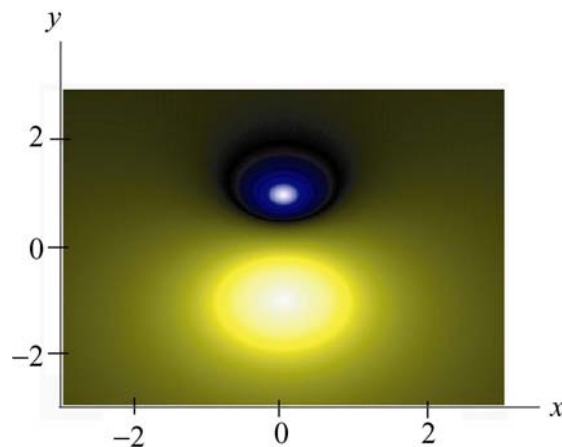
Various contour levels are shown in Figure 1.2.4, for  $d = 1$ , labeled by the value of the function at that level.



**Figure 1.2.4** A contour map in the  $xy$ -plane of the scalar field given by Eq. (1.2.3).

## 2. Color-Coding

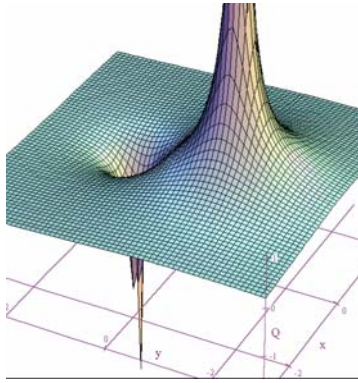
Another way we can represent the values of the scalar field is by color-coding in two dimensions for a fixed value of the third. This was the scheme used for illustrating the temperature fields in Figures 1.2.1 and 1.2.2. In Figure 1.2.5 a similar map is shown for the scalar field  $\phi(x, y, 0)$ . Different values of  $\phi(x, y, 0)$  are characterized by different colors in the map.



**Figure 1.2.5** A color-coded map in the  $xy$ -plane of the scalar field given by Eq. (1.2.3).

### 3. Relief Maps

A third way to represent a scalar field is to fix one of the dimensions, and then plot the value of the function as a height versus the remaining spatial coordinates, say  $x$  and  $y$ , that is, as a relief map. Figure 1.2.6 shows such a map for the same function  $\phi(x, y, 0)$ .

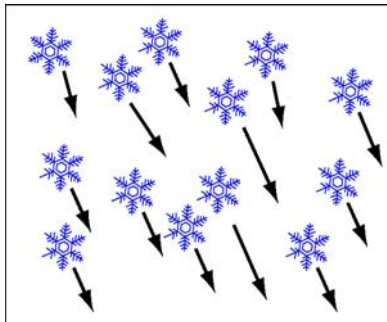


**Figure 1.2.6** A relief map of the scalar field given by Eq. (1.2.3).

### 1.3 Vector Fields

A vector is a quantity which has both a magnitude and a direction in space. Vectors are used to describe physical quantities such as velocity, momentum, acceleration and force, associated with an object. However, when we try to describe a system which consists of a large number of objects (e.g., moving water, snow, rain,...) we need to assign a vector to each individual object.

As an example, let's consider falling snowflakes, as shown in Figure 1.3.1. As snow falls, each snowflake moves in a specific direction. The motion of the snowflakes can be analyzed by taking a series of photographs. At any instant in time, we can assign, to each snowflake, a velocity vector which characterizes its movement.



**Figure 1.3.1** Falling snow.

The falling snow is an example of a collection of discrete bodies. On the other hand, if we try to analyze the motion of continuous bodies such as fluids, a velocity vector then needs to be assigned to every point in the fluid at any instant in time. Each vector describes the direction and magnitude of the velocity at a particular point and time. The collection of all the velocity vectors is called the velocity vector field. An important distinction between a vector field and a scalar field is that the former contains information about both the direction and the magnitude at every point in space, while only a single variable is specified for the latter. An example of a system of continuous bodies is air flow.

## 1.4 Fluid Flow

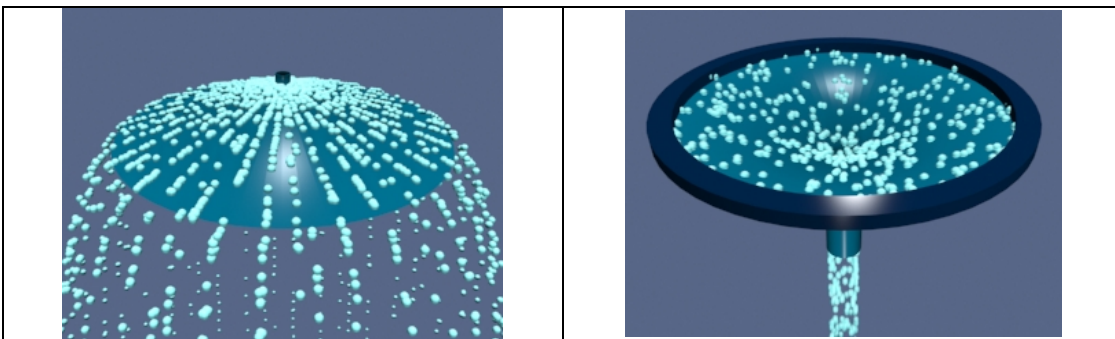
### Animation 1.1: Sources and Sinks

In general, a vector field  $\vec{F}(x, y, z)$  can be written as

$$\vec{F}(x, y, z) = F_x(x, y, z)\hat{i} + F_y(x, y, z)\hat{j} + F_z(x, y, z)\hat{k} \quad (1.4.1)$$

where the components are scalar fields. Below we use fluids to examine the properties associated with a vector field since fluid flows are the easiest vector fields to visualize.

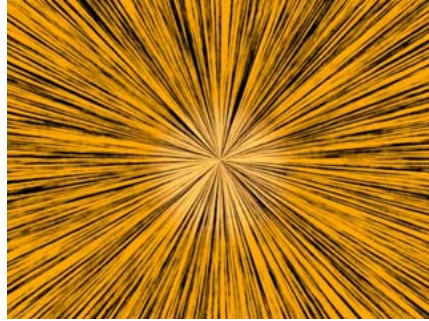
In Figure 1.4.1 we show physical examples of a fluid flow field, where we represent the fluid by a finite number of particles to show the structure of the flow. In Figure 1.4.1(a), particles (fluid elements) appear at the center of a cone (a “source”) and then flow downward under the effect of gravity. That is, we create particles at the origin, and they subsequently flow away from their creation point. We also call this a diverging flow, since the particles appear to “diverge” from the creation point. Figure 1.4.1(b) is the converse of this, a converging flow, or a “sink” of particles.



**Figure 1.4.1** (a) An example of a source of particles and the flow associated with a source, (b) An example of a sink of particles and the flow associated with a sink.

Another representation of a diverging flow is in depicted in Figure 1.4.2.

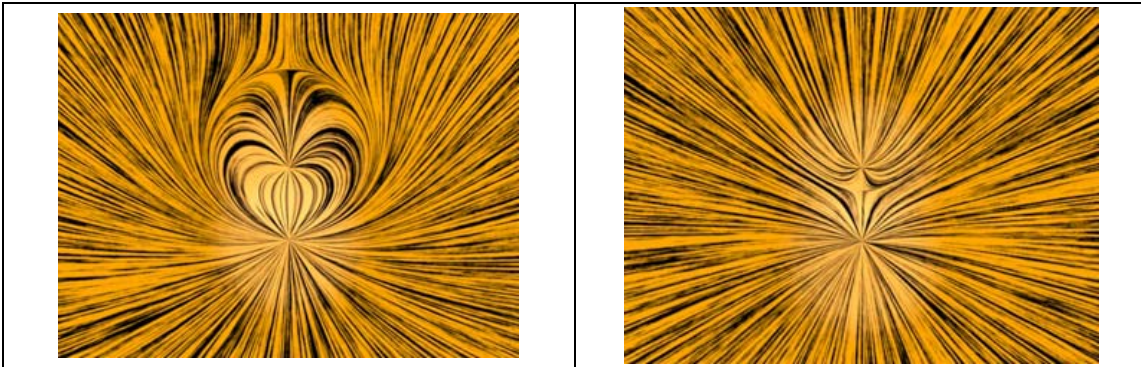




**Figure 1.4.2** Representing the flow field associated with a source using textures.

Here the direction of the flow is represented by a texture pattern in which the direction of correlation in the texture is along the field direction.

Figure 1.4.3(a) shows a source next to a sink of lesser magnitude, and Figure 1.4.3(b) shows two sources of unequal strength.



**Figure 1.4.3** The flow fields associated with (a) a source (lower) and a sink (upper) where the sink is smaller than the source, and (b) two sources of unequal strength.

Finally, in Figure 1.4.4, we illustrate a constant downward flow interacting with a diverging flow (source). The diverging flow is able to make some headway “upwards” against the downward constant flow, but eventually turns and flows downward, overwhelmed by the strength of the “downward” flow.



**Figure 1.4.4** A constant downward flow interacting with a diverging flow (source).  
 In the language of vector calculus, we represent the flow field of a fluid by

$$\vec{v} = v_x \hat{i} + v_y \hat{j} + v_z \hat{k} \quad (1.4.2)$$

A point  $(x, y, z)$  is a source if the divergence of  $\vec{v}(x, y, z)$  is greater than zero. That is,

$$\nabla \cdot \vec{v}(x, y, z) = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} > 0 \quad (1.4.3)$$

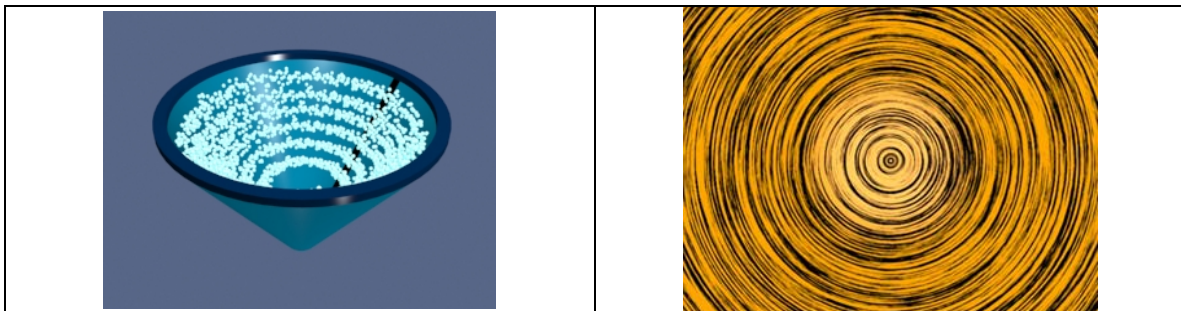
where

$$\nabla = \frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \quad (1.4.4)$$

is the del operator. On the other hand,  $(x, y, z)$  is a sink if the divergence of  $\vec{v}(x, y, z)$  is less than zero. When  $\nabla \cdot \vec{v}(x, y, z) = 0$ , then the point  $(x, y, z)$  is neither a source nor a sink. A fluid whose flow field has zero divergence is said to be *incompressible*.

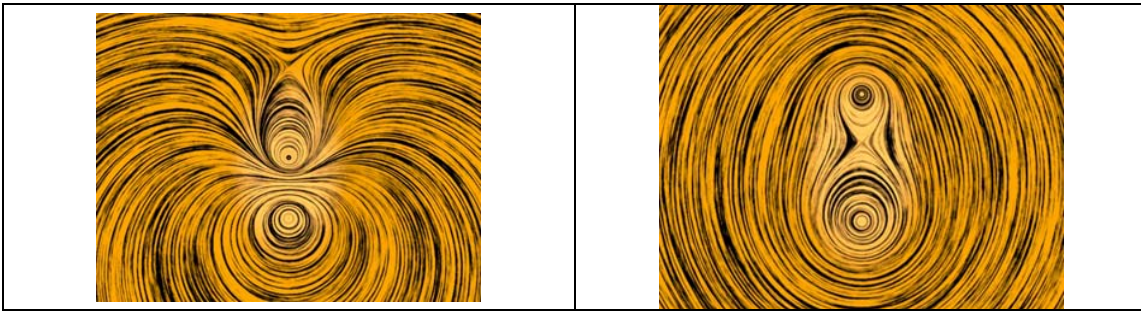
### Animation 1.2: Circulations

A flow field which is neither a source nor a sink may exhibit another class of behavior - circulation. In Figure 1.4.5(a) we show a physical example of a circulating flow field where particles are not created or destroyed (except at the beginning of the animation), but merely move in circles. The purely circulating flow can also be represented by textures, as shown in Figure 1.4.5(b).



**Figure 1.4.5** (a) An example of a circulating fluid. (b) Representing a circulating flow using textures.

A flow field can have more than one system of circulation centered about different points in space. In Figure 1.4.6(a) we show a flow field with two circulations. The flows are in opposite senses, and one of the circulations is stronger than the other. In Figure 1.4.6(b) we have the same situation, except that now the two circulations are in the same sense.



**Figure 1.4.6** A flow with two circulation centers with (a) opposite directions of circulation. (b) the same direction of circulation

In Figure 1.4.7, we show a constant downward flow interacting with a counter-clockwise circulating flow. The circulating flow is able to make some headway against the downward constant flow, but eventually is overwhelmed by the strength of the “downward” flow.



**Figure 1.4.7** A constant downward flow interacting with a counter-clockwise circulating flow.

In the language of vector calculus, the flows shown in Figures 1.4.5 through 1.4.7 are said to have a non-zero curl, but zero divergence. In contrast, the flows shown in Figures 1.4.2 through 1.4.4 have a zero curl (they do not move in circles) and a non-zero divergence (particles are created or destroyed).

Finally, in Figure 1.4.8, we show a fluid flow field that has both a circulation and a divergence (both the divergence and the curl of the vector field are non-zero). Any vector field can be written as the sum of a curl-free part (no circulation) and a divergence-free part (no source or sink). We will find in our study of electrostatics and magnetostatics that the electrostatic fields are curl free (e.g. they look like Figures 1.4.2 through 1.4.4) and the magnetic fields are divergence free (e.g. they look like Figures 1.4.5 and 1.4.6). Only when dealing with time-varying situations will we encounter electric fields that have both a divergence and a curl. Figure 1.4.8 depicts a field whose curl and divergence are non-vanishing. As far as we know even in time-varying situations magnetic fields always remain divergence-free. Therefore, magnetic fields will always look like the patterns shown in Figures 1.4.5 through 1.4.7.



**Figure 1.4.8** A flow field that has both a source (divergence) and a circulation (curl).

### 1.4.1 Relationship Between Fluid Fields and Electromagnetic Fields

Vector fields that represent fluid flow have an immediate physical interpretation: the vector at every point in space represents a direction of motion of a fluid element, and we can construct animations of those fields, as above, which show that motion. A more general vector field, for example the electric and magnetic fields discussed below, do not have that immediate physical interpretation of a flow field. There is no “flow” of a fluid along an electric field or magnetic field.

However, even though the vectors in electromagnetism do not represent fluid flow, we carry over many of the terms we use to describe fluid flow to describe electromagnetic fields as well. For example we will speak of the flux (flow) of the electric field through a surface. If we were talking about fluid flow, “flux” would have a well-defined physical meaning, in that the flux would be the amount of fluid flowing across a given surface per unit time. There is no such meaning when we talk about the flux of the electric field through a surface, but we still use the same term for it, as if we were talking about fluid flow. Similarly we will find that magnetic vector field exhibit patterns like those shown above for circulating flows, and we will sometimes talk about the circulation of magnetic fields. But there is no fluid circulating along the magnetic field direction.

We use much of the terminology of fluid flow to describe electromagnetic fields because it helps us understand the structure of electromagnetic fields intuitively. However, we must always be aware that the analogy is limited.

## 1.5 Gravitational Field

The gravitational field of the Earth is another example of a vector field which can be used to describe the interaction between a massive object and the Earth. According to Newton’s universal law of gravitation, the gravitational force between two masses  $m$  and  $M$  is given by

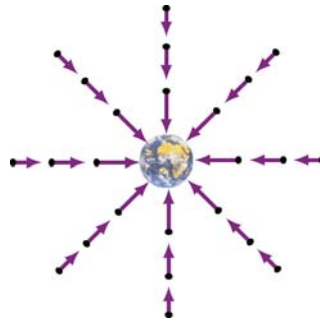
$$\vec{\mathbf{F}}_g = -G \frac{Mm}{r^2} \hat{\mathbf{r}} \quad (1.5.1)$$

where  $r$  is the distance between the two masses and  $\hat{\mathbf{r}}$  is the unit vector located at the position of  $m$  that points from  $M$  towards  $m$ . The constant of proportionality is the gravitational constant  $G = 6.67 \times 10^{-11} \text{ N} \cdot \text{m}^2 / \text{kg}^2$ . Notice that the force is always attractive, with its magnitude being proportional to the inverse square of the distance between the masses.

As an example, if  $M$  is the mass of the Earth, the gravitational field  $\vec{\mathbf{g}}$  at a point  $P$  in space, defined as the gravitational force per unit mass, can be written as

$$\vec{\mathbf{g}} = \lim_{m \rightarrow 0} \frac{\vec{\mathbf{F}}_g}{m} = -G \frac{M}{r^2} \hat{\mathbf{r}} \quad (1.5.2)$$

From the above expression, we see that the field is radial and points toward the center of the Earth, as shown in Figure 1.5.1.

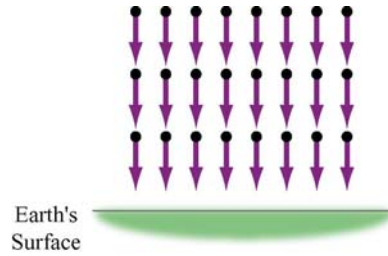


**Figure 1.5.1** Gravitational field of the Earth.

Near the Earth's surface, the gravitational field  $\vec{\mathbf{g}}$  is approximately constant:  $\vec{\mathbf{g}} = -g\hat{\mathbf{r}}$ , where

$$g = G \frac{M}{R_E^2} \approx 9.8 \text{ m/s}^2 \quad (1.5.3)$$

and  $R_E$  is the radius of Earth. The gravitational field near the Earth's surface is depicted in Figure 1.5.2.



**Figure 1.5.2** Uniform gravitational field near the surface of the Earth.

Notice that a mass in a constant gravitational field does not necessarily move in the direction of the field. This is true only when its initial velocity is in the same direction as the field. On the other hand, if the initial velocity has a component perpendicular to the gravitational field, the trajectory will be parabolic.

## 1.6 Electric Fields

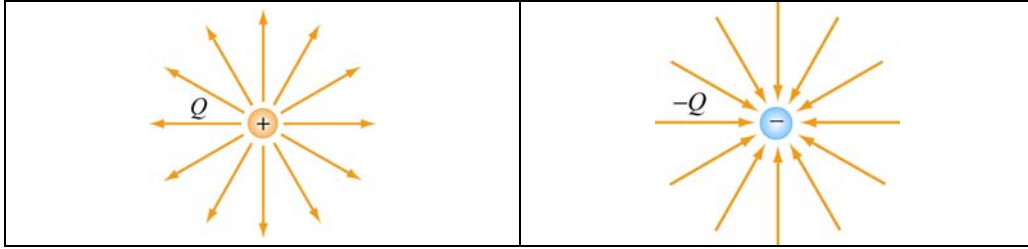
The interaction between electric charges at rest is called the electrostatic force. However, unlike mass in gravitational force, there are two types of electric charge: positive and negative. Electrostatic force between charges falls off as the inverse square of their distance of separation, and can be either attractive or repulsive. Electric charges exert forces on each other in a manner that is analogous to gravitation. Consider an object which has charge  $Q$ . A “test charge” that is placed at a point  $P$  a distance  $r$  from  $Q$  will experience a Coulomb force:

$$\vec{F}_e = k_e \frac{Qq}{r^2} \hat{r} \quad (1.6.1)$$

where  $\hat{r}$  is the unit vector that points from  $Q$  to  $q$ . The constant of proportionality  $k_e = 9.0 \times 10^9 \text{ N} \cdot \text{m}^2 / \text{C}^2$  is called the Coulomb constant. The electric field at  $P$  is defined as

$$\vec{E} = \lim_{q \rightarrow 0} \frac{\vec{F}_e}{q} = k_e \frac{Q}{r^2} \hat{r} \quad (1.6.2)$$

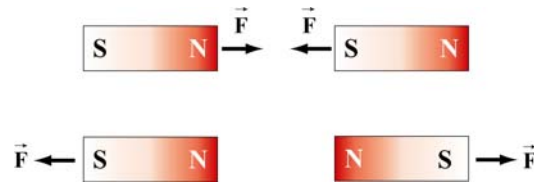
The SI unit of electric field is newtons/coulomb (N/C). If  $Q$  is positive, its electric field points radially away from the charge; on the other hand, the field points radially inward if  $Q$  is negative (Figure 1.6.1). In terms of the field concept, we may say that the charge  $Q$  creates an electric field  $\vec{E}$  which exerts a force  $\vec{F}_e = q\vec{E}$  on  $q$ .



**Figure 1.6.1** Electric field for positive and negative charges

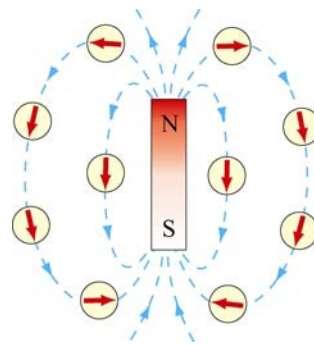
## 1.7 Magnetic Field

Magnetic field is another example of a vector field. The most familiar source of magnetic fields is a bar magnet. One end of the bar magnet is called the North pole and the other, the South pole. Like poles repel while opposite poles attract (Figure 1.7.1).



**Figure 1.7.1** Magnets attracting and repelling

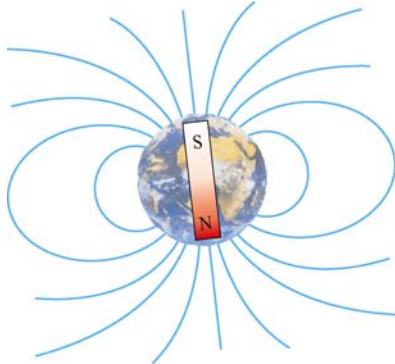
If we place some compasses near a bar magnet, the needles will align themselves along the direction of the magnetic field, as shown in Figure 1.7.2.



**Figure 1.7.2** Magnetic field of a bar magnet

The observation can be explained as follows: A magnetic compass consists of a tiny bar magnet that can rotate freely about a pivot point passing through the center of the magnet. When a compass is placed near a bar magnet which produces an external magnetic field, it experiences a torque which tends to align the north pole of the compass with the external magnetic field.

The Earth's magnetic field behaves as if there were a bar magnet in it (Figure 1.7.3). Note that the south pole of the magnet is located in the northern hemisphere.



**Figure 1.7.3** Magnetic field of the Earth

## 1.8 Representations of a Vector Field

How do we represent vector fields? Since there is much more information (magnitude and direction) in a vector field, our visualizations are correspondingly more complex when compared to the representations of scalar fields.

Let us introduce an analytic form for a vector field and discuss the various ways that we represent it. Let

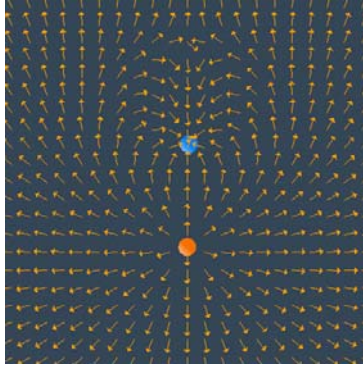
$$\vec{\mathbf{E}}(x, y, z) = \frac{x\hat{\mathbf{i}} + (y+d)\hat{\mathbf{j}} + z\hat{\mathbf{k}}}{[x^2 + (y+d)^2 + z^2]^{3/2}} - \frac{1}{3} \frac{x\hat{\mathbf{i}} + (y-d)\hat{\mathbf{j}} + z\hat{\mathbf{k}}}{[x^2 + (y-d)^2 + z^2]^{3/2}} \quad (1.8.1)$$

This field is proportional to the electric field of two point charges of opposite signs, with the magnitude of the positive charge three times that of the negative charge. The positive charge is located at  $(0, -d, 0)$  and the negative charge is located at  $(0, d, 0)$ . We discuss how this field is calculated in Section 2.7.

### 1.8.1 Vector Field Representation

Figure 1.8.1 is an example of a “vector field” representation of Eq. (1.8.1), in the plane where  $z = 0$ . We show the charges that would produce this field if it were an electric field, one positive (the orange charge) and one negative (the blue charge). We will always use this color scheme to represent positive and negative charges.





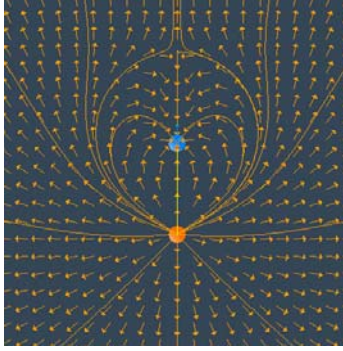
**Figure 1.8.1** A “vector field” representation of the field of two point charges, one negative and one positive, with the magnitude of the positive charge three times that of the negative charge. In the applet linked to this figure, one can vary the magnitude of the charges and the spacing of the vector field grid, and move the charges about.

In the vector field representation, we put arrows representing the field direction on a rectangular grid. The direction of the arrow at a given location represents the direction of the vector field at that point. In many cases, we also make the length of the vector proportional to the magnitude of the vector field at that point. But we also may show only the direction with the vectors (that is make all vectors the same length), and color-code the arrows according to the magnitude of the vector. Or we may not give any information about the magnitude of the field at all, but just use the arrows on the grid to indicate the direction of the field at that point.

Figure 1.8.1 is an example of the latter situation. That is, we use the arrows on the vector field grid to simply indicate the direction of the field, with no indication of the magnitude of the field, either by the length of the arrows or their color. Note that the arrows point away from the positive charge (the positive charge is a “source” for electric field) and towards the negative charge (the negative charge is a “sink” for electric field).

## 1.8.2 Field Line Representation

There are other ways to represent a vector field. One of the most common is to draw “field lines.” Faraday called the field lines for electric field “lines of force.” To draw a field line, start out at any point in space and move a very short distance in the direction of the local vector field, drawing a line as you do so. After that short distance, stop, find the new direction of the local vector field at the point where you stopped, and begin moving again in that new direction. Continue this process indefinitely. Thereby you construct a line in space that is everywhere tangent to the local vector field. If you do this for different starting points, you can draw a set of field lines that give a good representation of the properties of the vector field. Figure 1.8.2 below is an example of a field line representation for the same two charges we used in Figure 1.8.1.



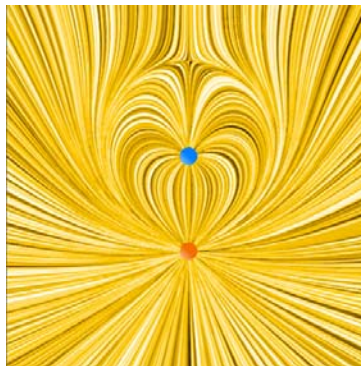
The field lines are everywhere tangent to the local field direction.

In summary, the field lines are a representation of the collection of vectors that constitute the field, and they are drawn according to the following rules:

- (1) The direction of the field line at any point in space is tangent to the field at that point.
- (2) The field lines never cross each other, otherwise there would be two different field directions at the point of intersection.

### 1.8.3 Grass Seeds and Iron Filings Representations

The final representation of vector fields is the “grass seeds” representation or the “iron filings” representation. For an electric field, this name derives from the fact that if you scatter grass seeds in a strong electric field, they will orient themselves with the long axis of the seed parallel to the local field direction. They thus provide a dense sampling of the shape of the field. Figure 1.8.4 is a “grass seeds” representation of the electric field for the same two charges in Figures 1.8.1 and 1.8.2.



**Figure 1.8.4:** A “grass seeds” representation of the electric field that we considered in Figures 1.8.1 and 1.8.2. In the applet linked to this figure, one can generate “grass seeds” representations for different amounts of charge and different positions.

The local field direction is in the direction in which the texture pattern in this figure is correlated. This “grass seeds” representation gives by far the most information about the spatial structure of the field.

We will also use this technique to represent magnetic fields, but when used to represent magnetic fields we call it the “iron filings” representation. This name derives from the fact that if you scatter iron filings in a strong magnetic field, they will orient themselves with their long axis parallel to the local field direction. They thus provide a dense sampling of the shape of the magnetic field.

A frequent question from the student new to electromagnetism is “What is between the field lines?” Figures 1.8.2 and 1.8.4 make the answer to that question clear. What is between the field lines are more field lines that we have chosen not to draw. The field itself is a continuous feature of the space between the charges.

### 1.8.4 Motion of Electric and Magnetic Field Lines

In this course we will show the spatial structure of electromagnetic fields using all of the methods discussed above. In addition, for the field line and the grass seeds and iron filings representation, we will frequently show the time evolution of the fields. We do this by having the field lines and the grass seed patterns or iron filings patterns move in the direction of the energy flow in the electromagnetic field at a given point in space. The flow is in the direction of  $\vec{E} \times \vec{B}$ , the cross product of the electric field  $\vec{E}$  and the magnetic field  $\vec{B}$ , and is perpendicular to both  $\vec{E}$  and  $\vec{B}$ . This is very different from our representation of fluid flow fields above, where the direction of the flow is in the same direction as the velocity field itself. We will discuss the concept on energy flow in electromagnetic fields toward the end of the course.

We adopt this representation for time-changing electromagnetic fields because these fields can both support the flow of energy and can store energy as well. We will discuss quantitatively how to compute this energy flow later, when we discuss the Poynting vector in Chapter 13. For now we simply note that when we animate the motion of the field line or grass seeds or iron filings representations, the direction of the pattern motion indicates the direction in which energy in the electromagnetic field is flowing.

## 1.9 Summary

In this chapter, we have discussed the concept of fields. A scalar field  $T(x, y, z)$  is a function on all the coordinates of space. Examples of a scalar field include temperature and pressure. On the other hand, a vector field  $\vec{F}(x, y, z)$  is a vector each of whose components is a scalar field. A vector field  $\vec{F}(x, y, z)$  has both magnitude and direction at every point  $(x, y, z)$  in space. Gravitational, electric and magnetic fields are all examples of vector fields.

## 1.10 Solved Problems

### 1.10.1 Vector Fields

Make a plot of the following vector fields:

(a)  $\vec{v} = 3\hat{i} - 5\hat{j}$

This is an example of a constant vector field in two dimensions. The plot is depicted in Figure 1.10.1:

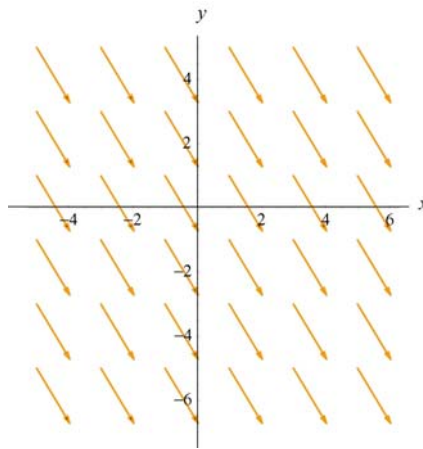


Figure 1.10.1

(b)  $\vec{v} = \vec{r}$

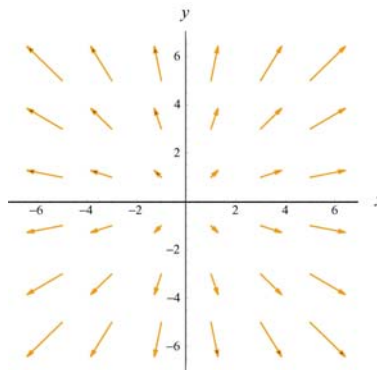


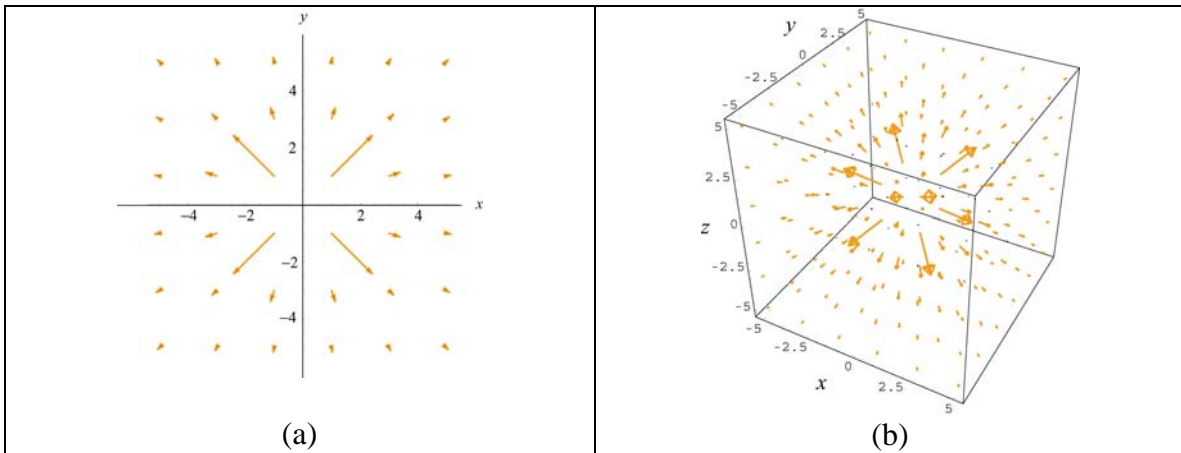
Figure 1.10.2

(c)  $\vec{v} = \frac{\hat{r}}{r^2}$

In two dimensions, using the Cartesian coordinates where  $\vec{r} = x\hat{i} + y\hat{j}$ ,  $\vec{v}$  can be written as

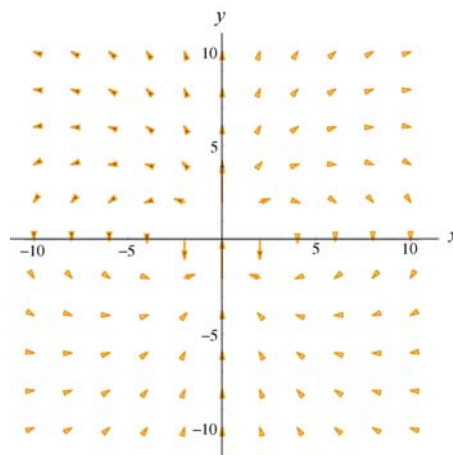
$$\vec{v} = \frac{\hat{r}}{r^2} = \frac{\vec{r}}{r^3} = \frac{x\hat{i} + y\hat{j}}{(x^2 + y^2)^{3/2}}$$

The plot is shown in Figure 1.10.3(a). Both the gravitational field of the Earth  $\vec{g}$  and the electric field  $\vec{E}$  due to a point charge have the same characteristic behavior as  $\vec{v}$ . In three dimensions where  $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ , the plot looks like that shown in Figure 1.10.3(b).



**Figure 1.10.3**

(d) 
$$\vec{v} = \frac{3xy}{r^5}\hat{i} + \frac{2y^2 - x^2}{r^5}\hat{j}$$



**Figure 1.10.4**

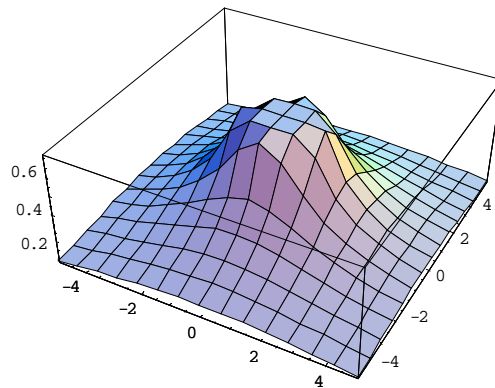
The plot is characteristic of the electric field due to a point electric dipole located at the origin.

## 1.10.2 Scalar Fields

Make a plot of the following scalar functions in two dimensions:

(a)  $f(r) = \frac{1}{r}$

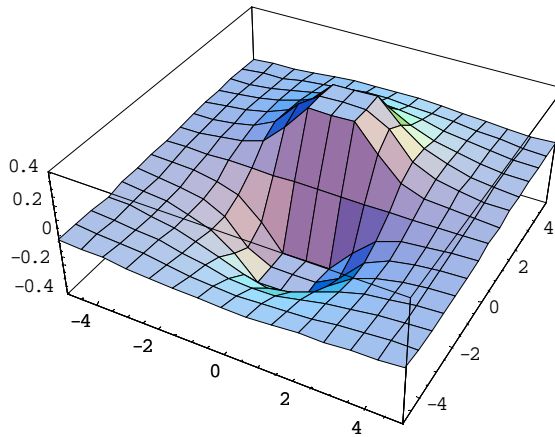
In two dimensions, we may write  $r = \sqrt{x^2 + y^2}$ .



**Figure 1.10.5**

Figure 1.10.5 can be used to represent the electric potential due to a point charge located at the origin. Notice that the mesh size has been adjusted so that the singularity at  $r = 0$  is not shown.

(b)  $f(x, y) = \frac{1}{\sqrt{x^2 + (y-1)^2}} - \frac{1}{\sqrt{x^2 + (y+1)^2}}$



**Figure 1.10.6**

This plot represents the potential due to a dipole with the positive charge located

at  $y = 1$  and the negative charge at  $y = -1$ . Again, singularities at  $(x, y) = (0, \pm 1)$  are not shown.

## 1.11 Additional Problems

### 1.11.1 Plotting Vector Fields

Plot the following vector fields:

$$\begin{array}{llll}
 \text{(a)} \ y\hat{\mathbf{i}} - x\hat{\mathbf{j}} & \text{(b)} \ \frac{1}{\sqrt{2}}(\hat{\mathbf{i}} - \hat{\mathbf{j}}) & \text{(c)} \ \frac{x\hat{\mathbf{i}} + y\hat{\mathbf{j}}}{\sqrt{2}} & \text{(d)} \ 2y\hat{\mathbf{i}} \quad \text{(e)} \ x^2\hat{\mathbf{i}} + y^2\hat{\mathbf{j}} \\
 \text{(f)} \ \frac{y\hat{\mathbf{i}} - x\hat{\mathbf{j}}}{\sqrt{x^2 + y^2}} & \text{(g)} \ xy\hat{\mathbf{i}} - x\hat{\mathbf{j}} & \text{(h)} \ \cos x\hat{\mathbf{i}} + \sin y\hat{\mathbf{j}} & 
 \end{array}$$

### 1.11.2 Position Vector in Spherical Coordinates

In spherical coordinates (see Figure 1.2.3), show that the position vector can be written as

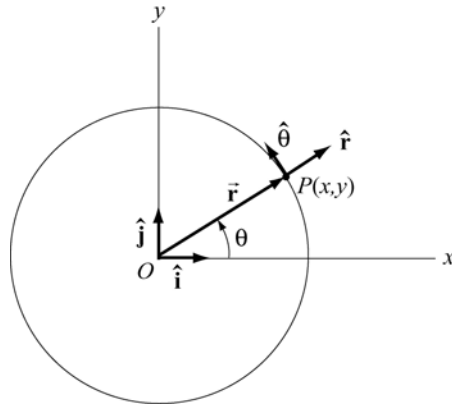
$$\vec{\mathbf{r}} = r \sin \theta \cos \phi \hat{\mathbf{i}} + r \sin \theta \sin \phi \hat{\mathbf{j}} + r \cos \theta \hat{\mathbf{k}}$$

### 1.11.3 Electric Field

A charge  $+1$  is situated at the point  $(-1, 0, 0)$  and a charge  $-1$  is situated at the point  $(1, 0, 0)$ . Find the electric field of these two charges at an arbitrary point  $(0, y, 0)$  on the  $y$ -axis.

### 1.11.4 An Object Moving in a Circle

A particle moves in a circular path of radius  $r$  in the  $xy$ -plane with a constant angular speed  $\omega = d\theta/dt$ . At some instant  $t$ , the particle is at  $P$ , as shown in Figure 1.11.1.



**Figure 1.11.1**

- Write down the position vector  $\vec{r}(t)$ .
- Calculate the velocity and acceleration of the particle at  $P$ .
- Express the unit vectors  $\hat{r}$  and  $\hat{\theta}$  in polar coordinates in terms of the unit vectors  $\hat{i}$  and  $\hat{j}$  in Cartesian coordinates.

### 1.11.5 Vector Fields

- Find a vector field in two dimensions which points in the negative radial direction and has magnitude 1.
- Find a vector field in two dimensions which makes an angle of  $45^\circ$  with the  $x$ -axis and has a magnitude  $(x + y)^2$  at any point  $(x, y)$ .
- Find a vector field in two dimensions whose direction is tangential and whose magnitude at any point  $(x, y)$  is equal to its distance from the origin.
- Find a vector field in three dimensions which is in the positive radial direction and whose magnitude is 1.

### 1.11.6 Object Moving in Two Dimensions

An object moving in two dimensions has a position vector

$$\vec{r}(t) = a \sin \omega t \hat{i} + b \cos \omega t \hat{j}$$

where  $a$ ,  $b$  and  $\omega$  are constants.

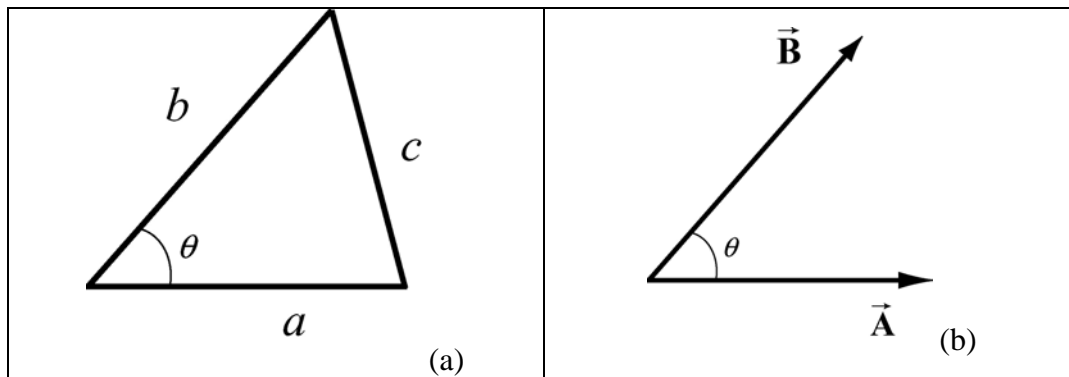
- How far is the object from the origin at time  $t$ ?



- (b) Find the velocity and acceleration as function of time for the object.
- (c) Show that the path of the object is elliptical.

### 1.11.7 Law of Cosines

Two sides of the triangle in Figure 1.11.2(a) form an angle  $\theta$ . The sides have lengths  $a$  and  $b$ .



**Figure 1.11.2** Law of cosines

The length of the side opposite  $\theta$  is given by the relation triangle identity

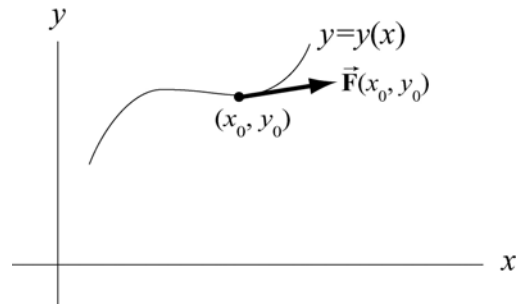
$$c^2 = a^2 + b^2 - 2ab \cos \theta .$$

Suppose we describe the two given sides of the triangles by the vectors  $\vec{\mathbf{A}}$  and  $\vec{\mathbf{B}}$ , with  $|\vec{\mathbf{A}}| = a$  and  $|\vec{\mathbf{B}}| = b$ , as shown in Figure 1.11.2(b)

- (a) What is the geometric meaning of the vector  $\vec{\mathbf{C}} = \vec{\mathbf{B}} - \vec{\mathbf{A}}$  ?
- (b) Show that the magnitude of  $\vec{\mathbf{C}}$  is equal to the length of the opposite side of the triangle shown in Figure 1.11.2(a), that is,  $|\vec{\mathbf{C}}| = c$ .

### 1.11.8 Field Lines

A curve  $y = y(x)$  is called a field line of the vector field  $\vec{\mathbf{F}}(x, y)$  if at every point  $(x_0, y_0)$  on the curve,  $\vec{\mathbf{F}}(x_0, y_0)$  is tangent to the curve (see Figure 1.11.3).



**Figure 1.11.3**

Show that the field lines  $y = y(x)$  of a vector field  $\vec{F}(x, y) = F_x(x, y)\hat{i} + F_y(x, y)\hat{j}$  represent the solutions of the differential equation

$$\frac{dy}{dx} = \frac{F_y(x, y)}{F_x(x, y)}$$