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18.01 Single Variable Calculus
Fall 2006

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18.01 Problem Set 8A

Due Friday 12/08/06, 1:55 pm

8A is the first half of Problem Set 8, all of which is due a week after Exam 4. (The second half, 8B, will be issued at the exam.) Even though it won't be collected until later, you should do 8A before the exam, to prepare for it.

Part I (15 points)

Lecture 30. Fri. Nov. 17 Parametric equations; arclength. Surface area.

Read 17.1, 7.5, 7.6 Work: 4E-2, 3, 8; 4F-1d, 4, 5, 8; 4G-2, 5.

If a curve is given by $x = x(t)$, $y = y(t)$, to find its arclength, use ds in the form

$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt,$$

and integrate from start to finish: from $t = t_0$ to $t = t_1$.

Lecture 31. Tues. Nov. 21 Polar coordinates; area in polar coordinates.

Read: 16.1, (16.2 lightly, for the pictures), 16.3 to top p.570, 16.5 to middle p.581

Work: 4H-1bfg; 4H-2a,3f; 4I-2,3

Lecture 32. Tues. Nov 28. Continuation and review.

Lecture 33. Thurs. Nov 30. **Exam 4, 1:05-1:55**, lectures 25–32.

Part II (30 points)

Directions: Attempt to solve *each part* of each problem yourself. If you collaborate, solutions must be written up independently. It is illegal to consult materials from previous semesters. With each problem is the day it can be done.

0. (not until due date; 3 pts) Write the names of all the people you consulted or with whom you collaborated and the resources you used, or say “none” or “no consultation”. (See full explanation on PS1).

1. (Lec 30, 7 pts: 3 + 1 + 3)

a) Find the algebraic equation in x and y for the curve

$$x = a \cos^k t, \quad y = a \sin^k t.$$

Draw the portion of the curve $0 \leq t \leq \pi/2$ in the three cases $k = 1$, $k = 2$, $k = 3$.

b) Without calculation, find the arclength in the cases $k = 1$ and $k = 2$.

c) Find a definite integral formula for the length of the curve for general k . Then evaluate the integral in the three cases $k = 1$, $k = 2$, and $k = 3$. (Your answer in the first two cases should match what you found in part (b), but the calculation takes more time.)

2. (Lec 30, 9 pts: 3 + 1 + 3 + 2) The hyperbolic sine and cosine are defined by

$$\cosh x = \frac{e^x + e^{-x}}{2}, \quad \sinh x = \frac{e^x - e^{-x}}{2}$$

a) Show that

i) $\frac{d}{dx} \sinh x = \cosh x$ and $\frac{d}{dx} \cosh x = \sinh x$.

ii) $\cosh^2 x = 1 + \sinh^2 x$ and

iii) $\cosh^2 x = \frac{1 + \cosh 2x}{2}$

b) What curve is described parametrically by $x = \cosh t$, $y = \sinh t$? (Give the equation and its name.)

c) The curve $y = \cosh x$ is known as a *catenary*. It is the curve formed by a chain whose two ends are held at the same height.

i) Sketch the curve

ii) Find its arclength from the lowest point to the point $(x_1, \cosh x_1)$ for a fixed $x_1 > 0$.

d) Find the area of the surface of revolution formed by revolving the portion of the curve from part (c) around the x -axis. This surface is known as a catenoid. It is interesting because it is the surface of least area connecting the two circles that form its edges. If you dip two circles of wire in a soap solution, then (with some coaxing) a soap film will form in this shape. In general, the soap films try to span a frame of wires with a surface with the least area possible.

3. (Lec 31, 11 pts: 3 + 2 + 2 + 4) Area in polar coordinates.

a) Find the area of the right triangle with vertices at $(x, y) = (0, 0)$, $(x, y) = (a, 0)$ and at $(x, y) = (a, h)$, using polar coordinates. (One of the less convenient ways to find the area of a triangle.)

b) Find the equation in (x, y) coordinates for the curve $r = 1/(1 + \sin \theta)$ and sketch it.

c) Find the area of the region $0 \leq r \leq 1/(1 + \sin \theta)$, $0 \leq \theta \leq \pi$, using the *rectangular* coordinate formula you found in part (b).

d) Find the area of the region in part (c) using polar coordinates. One way to evaluate the integral is to change variables to $u = \theta - \pi/2$, and then use the half angle formula

$$1 + \cos u = 2 \cos^2(u/2)$$

This area was already computed in part (c), but the polar coordinate formula is still valuable because it gives the area of any sector, not just the one that is bounded by a horizontal line. The area swept out from the viewpoint of the focus of an ellipse, parabola, or hyperbola is related by Kepler's law to the speed of planets and comets.