

## CHAPTER 2

# The Simple Pendulum

### 2.1 INTRODUCTION

Our goals for this chapter are modest: we'd like to understand the dynamics of a pendulum. Why a pendulum? In part, because the dynamics of a majority of our multi-link robotics manipulators are simply the dynamics of a large number of coupled pendula. Also, the dynamics of a single pendulum are rich enough to introduce most of the concepts from nonlinear dynamics that we will use in this text, but tractable enough for us to (mostly) understand in the next few pages.

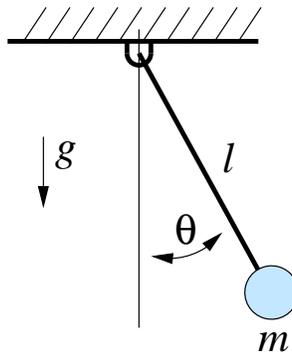


FIGURE 2.1 The Simple Pendulum

The Lagrangian derivation (e.g, [35]) of the equations of motion of the simple pendulum yields:

$$I\ddot{\theta}(t) + mgl \sin \theta(t) = Q,$$

where  $I$  is the moment of inertia, and  $I = ml^2$  for the simple pendulum. We'll consider the case where the generalized force,  $Q$ , models a damping torque (from friction) plus a control torque input,  $u(t)$ :

$$Q = -b\dot{\theta}(t) + u(t).$$

### 2.2 NONLINEAR DYNAMICS W/ A CONSTANT TORQUE

Let us first consider the dynamics of the pendulum if it is driven in a particular simple way: a torque which does not vary with time:

$$I\ddot{\theta} + b\dot{\theta} + mgl \sin \theta = u_0. \quad (2.1)$$

These are relatively simple equations, so we should be able to integrate them to obtain  $\theta(t)$  given  $\theta(0), \dot{\theta}(0)$ ... right? Although it is possible, integrating even the simplest case

( $b = u = 0$ ) involves elliptic integrals of the first kind; there is relatively little intuition to be gained here. If what we care about is the long-term behavior of the system, then we can investigate the system using a graphical solution method. These methods are described beautifully in a book by Steve Strogatz[83].

### 2.2.1 The Overdamped Pendulum

Let's start by studying a special case, when  $\frac{b}{l} \gg 1$ . This is the case of heavy damping - for instance if the pendulum was moving in molasses. In this case, the  $b$  term dominates the acceleration term, and we have:

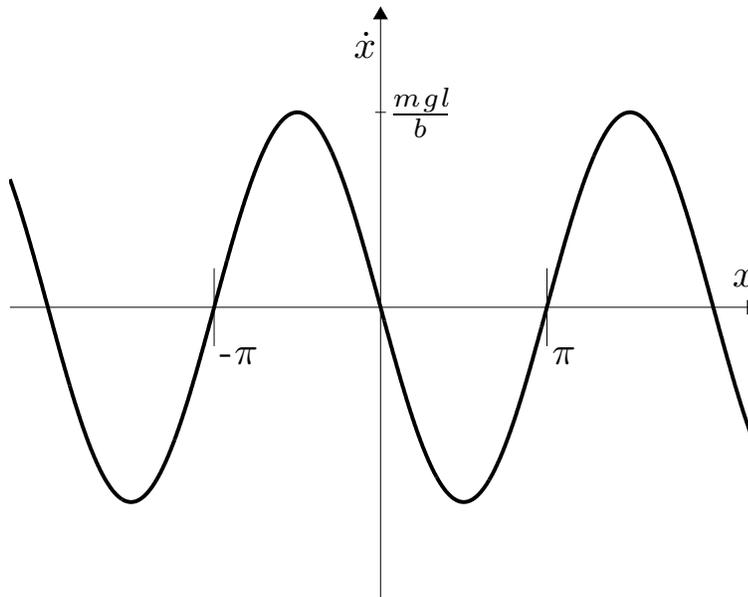
$$u_0 - mgl \sin \theta = I\ddot{\theta} + b\dot{\theta} \approx b\dot{\theta}.$$

In other words, in the case of heavy damping, the system looks approximately first-order. This is a general property of systems operating in fluids at very low Reynolds number.

I'd like to ignore one detail for a moment: the fact that  $\theta$  wraps around on itself every  $2\pi$ . To be clear, let's write the system without the wrap-around as:

$$b\dot{x} = u_0 - mgl \sin x. \quad (2.2)$$

Our goal is to understand the long-term behavior of this system: to find  $x(\infty)$  given  $x(0)$ . Let's start by plotting  $\dot{x}$  vs  $x$  for the case when  $u_0 = 0$ :



The first thing to notice is that the system has a number of *fixed points* or *steady states*, which occur whenever  $\dot{x} = 0$ . In this simple example, the zero-crossings are  $x^* = \{\dots, -\pi, 0, \pi, 2\pi, \dots\}$ . When the system is in one of these states, it will never leave that state. If the initial conditions are at a fixed point, we know that  $x(\infty)$  will be at the same fixed point.

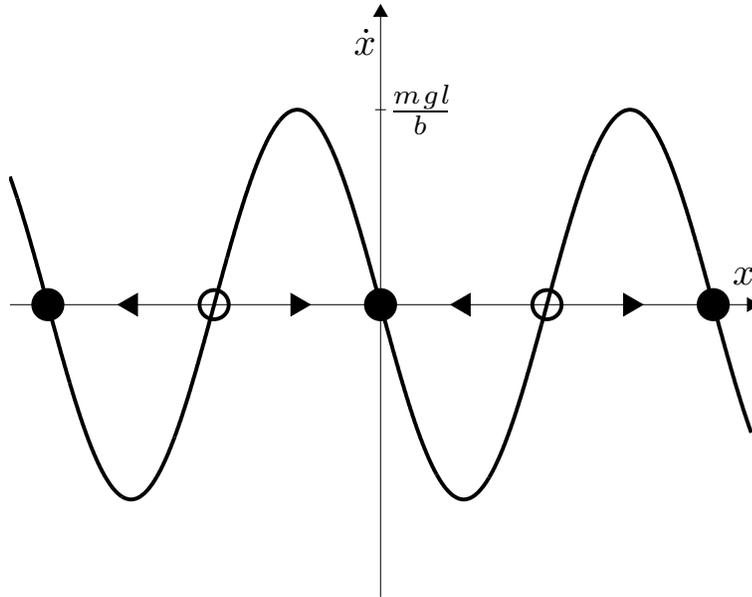
Next let's investigate the behavior of the system in the local vicinity of the fixed points. Examining the fixed point at  $x^* = \pi$ , if the system starts just to the right of the fixed point, then  $\dot{x}$  is positive, so the system will move away from the fixed point. If it starts to the left, then  $\dot{x}$  is negative, and the system will move away in the opposite direction. We'll call fixed-points which have this property *unstable*. If we look at the fixed point at  $x^* = 0$ , then the story is different: trajectories starting to the right or to the left will move back towards the fixed point. We will call this fixed point *locally stable*. More specifically, we'll distinguish between three types of local stability:

- **Locally stable in the sense of Lyapunov (i.s.L.).** A fixed point,  $x^*$  is locally stable i.s.L. if for every small  $\epsilon$ , I can produce a  $\delta$  such that if  $\|x(0) - x^*\| < \delta$  then  $\forall t$   $\|x(t) - x^*\| < \epsilon$ . In words, this means that for any ball of size  $\epsilon$  around the fixed point, I can create a ball of size  $\delta$  which guarantees that if the system is started inside the  $\delta$  ball then it will remain inside the  $\epsilon$  ball for all of time.
- **Locally asymptotically stable.** A fixed point is locally asymptotically stable if  $x(0) = x^* + \epsilon$  implies that  $x(\infty) = x^*$ .
- **Locally exponentially stable.** A fixed point is locally exponentially stable if  $x(0) = x^* + \epsilon$  implies that  $\|x(t) - x^*\| < Ce^{-\alpha t}$ , for some positive constants  $C$  and  $\alpha$ .

An initial condition near a fixed point that is stable in the sense of Lyapunov may never reach the fixed point (but it won't diverge), near an asymptotically stable fixed point will reach the fixed point as  $t \rightarrow \infty$ , and near an exponentially stable fixed point will reach the fixed point with a bounded rate. An exponentially stable fixed point is also an asymptotically stable fixed point, and an asymptotically stable fixed point is also stable i.s.L., but the converse of these is not necessarily true.

Our graph of  $\dot{x}$  vs.  $x$  can be used to convince ourselves of i.s.L. and asymptotic stability. Exponential stability could potentially be inferred if the function could be bounded by a negatively-sloped line through the fixed point, but this requires some care. I will graphically illustrate unstable fixed points with open circles and stable fixed points (i.s.L.) with filled circles. Next, we need to consider what happens to initial conditions which begin farther from the fixed points. If we think of the dynamics of the system as a flow on the  $x$ -axis, then we know that anytime  $\dot{x} > 0$ , the flow is moving to the right, and  $\dot{x} < 0$ , the flow is moving to the left. If we further annotate our graph with arrows indicating the direction of the flow, then the entire (long-term) system behavior becomes clear: For instance, we can see that any initial condition  $x(0) \in (\pi, \pi)$  will result in  $x(\infty) = 0$ . This region is called the *basin of attraction* of the fixed point at  $x^* = 0$ . Basins of attraction of two fixed points cannot overlap, and the manifold separating two basins of attraction is called the *separatrix*. Here the unstable fixed points, at  $x^* = \{\dots, -\pi, \pi, 3\pi, \dots\}$  form the separatrix between the basins of attraction of the stable fixed points.

As these plots demonstrate, the behavior of a first-order one dimensional system on a line is relatively constrained. The system will either monotonically approach a fixed-point or monotonically move toward  $\pm\infty$ . There are no other possibilities. Oscillations, for example, are impossible. Graphical analysis is a fantastic analysis tool for many first-order



nonlinear systems (not just pendula); as illustrated by the following example:

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EXAMPLE 2.1 Nonlinear autapse

Consider the following system:

$$\dot{x} + x = \tanh(wx) \quad (2.3)$$

It's convenient to note that  $\tanh(z) \approx z$  for small  $z$ . For  $w \leq 1$  the system has only a single fixed point. For  $w > 1$  the system has three fixed points : two stable and one unstable. These equations are not arbitrary - they are actually a model for one of the simplest neural networks, and one of the simplest model of persistent memory[71]. In the equation  $x$  models the firing rate of a single neuron, which has a feedback connection to itself.  $\tanh$  is the activation (sigmoidal) function of the neuron, and  $w$  is the weight of the synaptic feedback.

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One last piece of terminology. In the neuron example, and in many dynamical systems, the dynamics were parameterized; in this case by a single parameter,  $w$ . As we varied  $w$ , the fixed points of the system moved around. In fact, if we increase  $w$  through  $w = 1$ , something dramatic happens - the system goes from having one fixed point to having three fixed points. This is called a *bifurcation*. This particular bifurcation is called a pitchfork bifurcation. We often draw bifurcation diagrams which plot the fixed points of the system as a function of the parameters, with solid lines indicating stable fixed points and dashed lines indicating unstable fixed points, as seen in figure 2.2.

Our pendulum equations also have a (saddle-node) bifurcation when we change the constant torque input,  $u_0$ . This is the subject of exercise 1. Finally, let's return to the

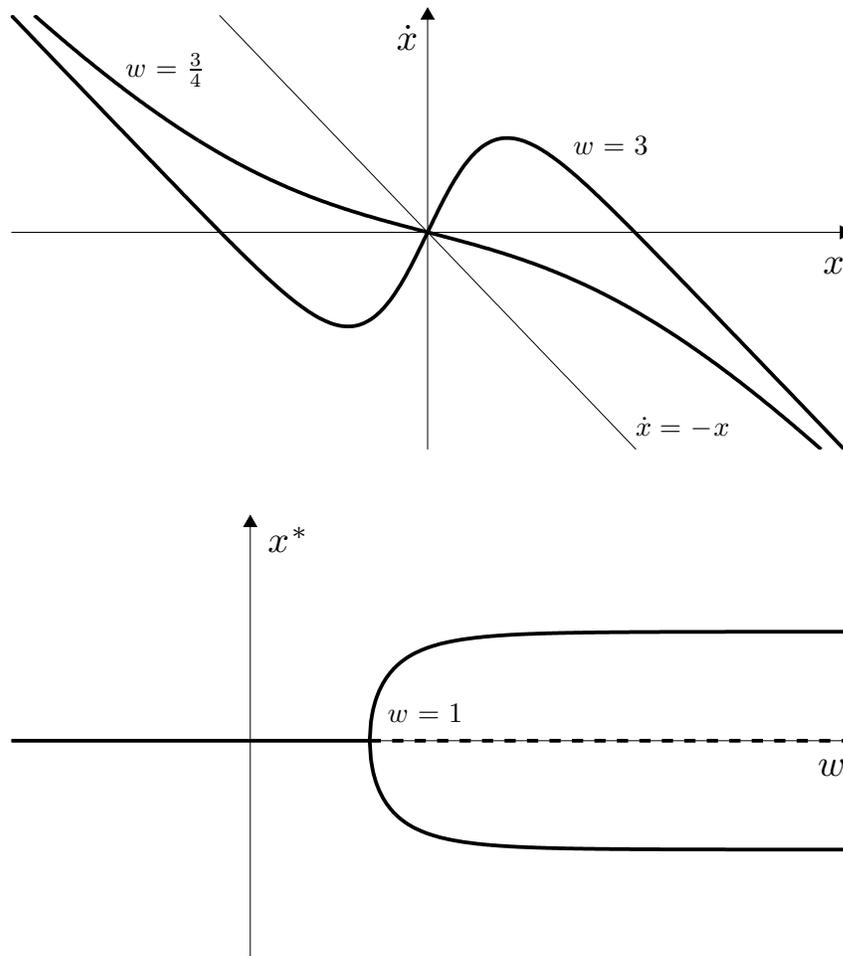


FIGURE 2.2 Bifurcation diagram of the nonlinear autopse.

original equations in  $\theta$ , instead of in  $x$ . Only one point to make: because of the wrap-around, this system will *appear* have oscillations. In fact, the graphical analysis reveals that the pendulum will turn forever whenever  $|u_0| > mgl$ .

### 2.2.2 The Undamped Pendulum w/ Zero Torque

Consider again the system

$$I\ddot{\theta} = u_0 - mgl \sin \theta - b\dot{\theta},$$

this time with  $b = 0$ . This time the system dynamics are truly second-order. We can always think of any second-order system as (coupled) first-order system with twice as many variables. Consider a general, autonomous (not dependent on time), second-order system,

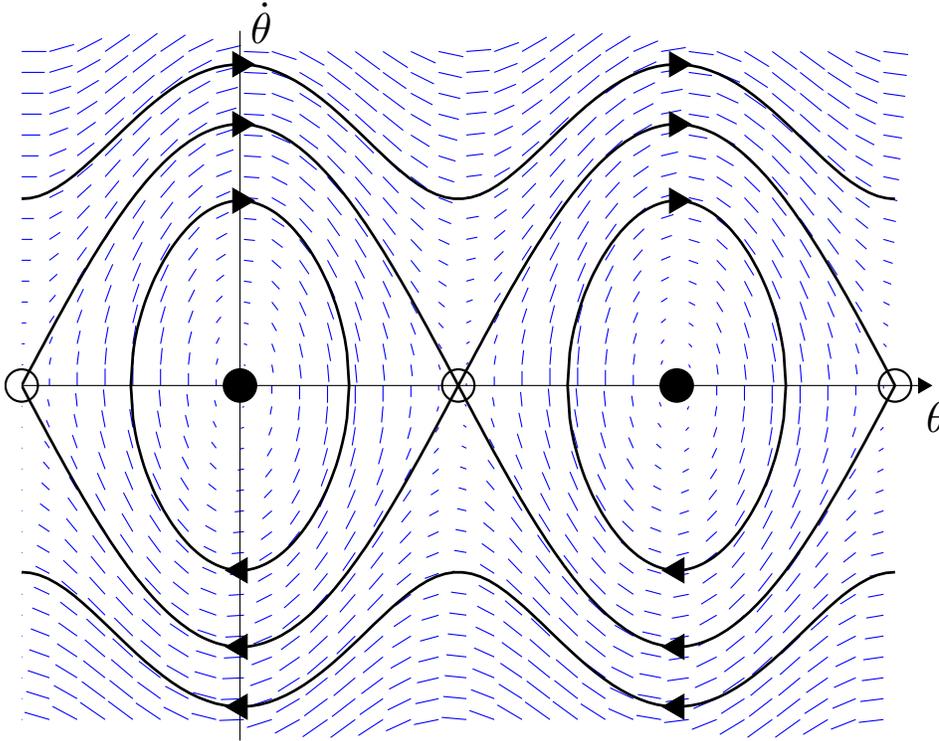
$$\ddot{q} = f(q, \dot{q}, u).$$

This system is equivalent to the two-dimensional first-order system

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= f(x_1, x_2, u),\end{aligned}$$

where  $x_1 = q$  and  $x_2 = \dot{q}$ . Therefore, the graphical depiction of this system is not a line, but a vector field where the vectors  $[\dot{x}_1, \dot{x}_2]^T$  are plotted over the domain  $(x_1, x_2)$ . This vector field is known as the *phase portrait* of the system.

In this section we restrict ourselves to the simplest case when  $u_0 = 0$ . Let's sketch the phase portrait. First sketch along the  $\theta$ -axis. The  $x$ -component of the vector field here is zero, the  $y$ -component is  $-mgl \sin \theta$ . As expected, we have fixed points at  $\pm\pi, \dots$  Now sketch the rest of the vector field. Can you tell me which fixed points are stable? Some of them are stable i.s.L., none are asymptotically stable.



#### Orbit Calculations.

Directly integrating the equations of motion is difficult, but at least for the case when  $u_0 = 0$ , we have some additional physical insight for this problem that we can take advantage of. The kinetic energy,  $T$ , and potential energy,  $U$ , of the pendulum are given by

$$T = \frac{1}{2}I\dot{\theta}^2, \quad U = -mgl \cos(\theta),$$

and the total energy is  $E(\theta, \dot{\theta}) = T(\dot{\theta}) + U(\theta)$ . The undamped pendulum is a conservative system: total energy is a constant over system trajectories. Using conservation of energy, we have:

$$\begin{aligned} E(\theta(t), \dot{\theta}(t)) &= E(\theta(0), \dot{\theta}(0)) = E \\ \frac{1}{2}I\dot{\theta}^2(t) - mgl \cos(\theta(t)) &= E \\ \dot{\theta}(t) &= \pm \sqrt{\frac{2}{I} [E + mgl \cos(\theta(t))]} \end{aligned}$$

This equation is valid (the squareroot evaluates to a real number) when  $\cos(\theta) > \cos(\theta_{max})$ , where

$$\theta_{max} = \begin{cases} \cos^{-1}\left(\frac{E}{mgl}\right), & E < mgl \\ \pi, & \text{otherwise.} \end{cases}$$

Furthermore, differentiating this equation with respect to time indeed results in the equations of motion.

#### Trajectory Calculations.

Solving for  $\theta(t)$  is a bit harder, because it cannot be accomplished using elementary functions. We begin the integration with

$$\begin{aligned} \frac{d\theta}{dt} &= \sqrt{\frac{2}{I} [E + mgl \cos(\theta(t))]} \\ \int_{\theta(0)}^{\theta(t)} \frac{d\theta}{\sqrt{\frac{2}{I} [E + mgl \cos(\theta(t))]}} &= \int_0^t dt' = t \end{aligned}$$

The integral on the left side of this equation is an (incomplete) elliptic integral of the first kind. Using the identity:

$$\cos(\theta) = 1 - 2 \sin^2\left(\frac{1}{2}\theta\right),$$

and manipulating, we have

$$t = \sqrt{\frac{I}{2(E + mgl)}} \int_{\theta(0)}^{\theta(t)} \frac{d\theta}{\sqrt{1 - k_1^2 \sin^2\left(\frac{\theta}{2}\right)}}, \quad \text{with } k_1 = \sqrt{\frac{2mgl}{E + mgl}}.$$

In terms of the incomplete elliptic integral function,

$$F(\phi, k) = \int_0^\phi \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}},$$

accomplished by a change of variables. If  $E \leq mgl$ , which is the case of closed-orbits, we use the following change of variables to ensure  $0 < k < 1$ :

$$\begin{aligned} \phi &= \sin^{-1} \left[ k_1 \sin \left( \frac{\theta}{2} \right) \right] \\ \cos(\phi) d\phi &= \frac{1}{2} k_1 \cos \left( \frac{\theta}{2} \right) d\theta = \frac{1}{2} k_1 \sqrt{1 - \frac{\sin^2(\phi)}{k_1^2}} d\theta \end{aligned}$$

we have

$$\begin{aligned}
 t &= \frac{1}{k_1} \sqrt{\frac{2I}{(E + mgl)}} \int_{\phi(0)}^{\phi(t)} \frac{d\phi}{\sqrt{1 - \sin^2(\phi)}} \frac{\cos(\phi)}{\sqrt{1 - \frac{\sin^2 \phi}{k_1^2}}} \\
 &= \sqrt{\frac{I}{mgl}} [F(\phi(t), k_2) - F(\phi(0), k_2)], \quad k_2 = \frac{1}{k_1}.
 \end{aligned}$$

The inverse of  $F$  is given by the Jacobi elliptic functions (sn,cn,...), yielding:

$$\begin{aligned}
 \sin(\phi(t)) &= \text{sn} \left( t \sqrt{\frac{mgl}{I}} + F(\phi(0), k_2), k_2 \right) \\
 \theta(t) &= 2 \sin^{-1} \left[ k_2 \text{sn} \left( t \sqrt{\frac{mgl}{I}} + F(\phi(0), k_2), k_2 \right) \right]
 \end{aligned}$$

The function sn used here can be evaluated in MATLAB by calling

$$\text{sn}(u, k) = \text{ellipj}(u, k^2).$$

The function  $F$  is not implemented in MATLAB, but implementations can be downloaded.. (note that  $F(0, k) = 0$ ).

For the open-orbit case,  $E > mgl$ , we use

$$\phi = \frac{\theta}{2}, \quad \frac{d\phi}{d\theta} = \frac{1}{2},$$

yielding

$$\begin{aligned}
 t &= \frac{2I}{E + mgl} \int_{\phi(0)}^{\phi(t)} \frac{d\phi}{\sqrt{1 - k_1^2 \sin^2(\phi)}} \\
 \theta(t) &= 2 \tan^{-1} \left[ \frac{\text{sn} \left( t \sqrt{\frac{E+mgl}{2I}} + F \left( \frac{\theta(0)}{2}, k_1 \right) \right)}{\text{cn} \left( t \sqrt{\frac{E+mgl}{2I}} + F \left( \frac{\theta(0)}{2}, k_1 \right) \right)} \right]
 \end{aligned}$$

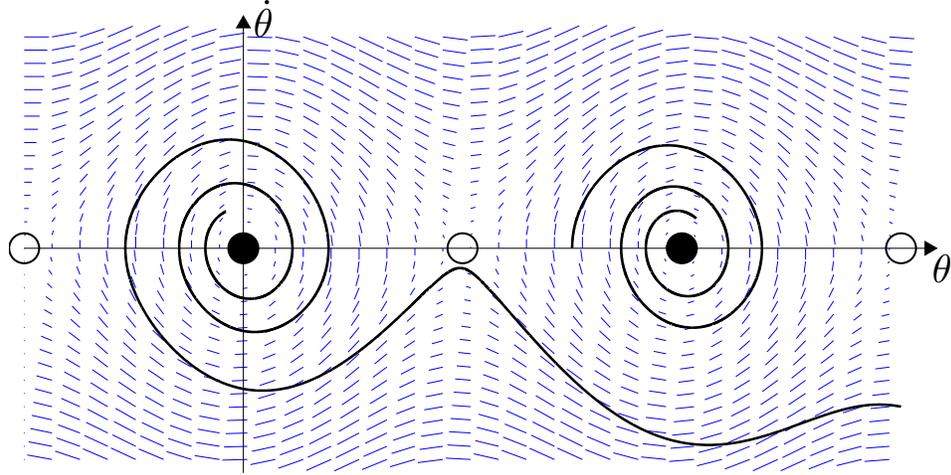
Notes: Use MATLAB's atan2 and unwrap to recover the complete trajectory.

### 2.2.3 The Undamped Pendulum w/ a Constant Torque

Now what happens if we add a constant torque? Fixed points come together, towards  $q = \frac{\pi}{2}, \frac{5\pi}{2}, \dots$ , until they disappear. Right fixed-point is unstable, left is stable.

### 2.2.4 The Damped Pendulum

Add damping back. You can still add torque to move the fixed points (in the same way).



Here's a thought exercise. If  $u$  is no longer a constant, but a function  $\pi(q, \dot{q})$ , then how would you choose  $\pi$  to stabilize the vertical position. Feedback linearization is the trivial solution, for example:

$$u = \pi(q, \dot{q}) = 2\frac{g}{l} \cos \theta.$$

But these plots we've been making tell a different story. How would you shape the natural dynamics - at each point pick a  $u$  from the stack of phase plots - to stabilize the vertical fixed point *with minimal torque effort*? We'll learn that soon.

### 2.3 THE TORQUE-LIMITED SIMPLE PENDULUM

The simple pendulum is fully actuated. Given enough torque, we can produce any number of control solutions to stabilize the originally unstable fixed point at the top (such as designing a feedback law to effectively invert gravity).

The problem begins to get interesting if we impose a torque-limit constraint,  $|u| \leq u_{max}$ . Looking at the phase portraits again, you can now visualize the control problem. Via feedback, you are allowed to change the direction of the vector field at each point, but only by a fixed amount. Clearly, if the maximum torque is small (smaller than  $mgl$ ), then there are some states which cannot be driven directly to the goal, but must pump up energy to reach the goal. Furthermore, if the torque-limit is too severe and the system has damping, then it may be impossible to swing up to the top. The existence of a solution, and number of pumps required to reach the top, is a non-trivial function of the initial conditions and the torque-limits.

Although this problem is still fully-actuated, its solution requires much of the same reasoning necessary for controller underactuated systems; this problem will be a work-horse for us as we introduce new algorithms throughout this book.

## PROBLEMS

**2.1.** *Bifurcation diagram of the simple pendulum.*

- (a) Sketch the bifurcation diagram by varying the continuous torque,  $u_0$ , in the *overdamped* simple pendulum described in Equation (2.2) over the range  $[-\frac{\pi}{2}, \frac{3\pi}{2}]$ . Carefully label the domain of your plot.
- (b) Sketch the bifurcation diagram of the underdamped pendulum over the same domain and range as in part (a).

**2.2. (CHALLENGE)** *The Simple Pendulum ODE.*

The chapter contained the closed-form solution for the undamped pendulum with zero torque.

- (a) Find the closed-form solution for the pendulum equations with a constant torque.
- (b) Find the closed-form solution for the pendulum equations with damping.
- (c) Find the closed-form solution for the pendulum equations with both damping and a constant torque.

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